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SERIES A
BIOLOGY

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APPLICATION OF MICROCARRIERS TO A RABBIT KIDNEY CELL LINE

Ç. Cırakoğlu[(1)], G. Omurtay[(2)], M. Arıkan[(2)]

Microcarriers (Cytodex 1) were applied to a rabbit kidney (RK) cell line.

We found that 1 mg/ml Cytodex1 was optimal for growth of rabbit kidney cells. Rabbit kidney cells were subcultered every seven days when they were grown as a monolayer in glass culture bottles. On the other hand RK cells grown on microcarriers were subcultered every fifteen days. Cytodex beads provide a large surface area for the cells and therefore cells grown on Cytodex 1 maintain excellent growth kinetics over long periods.

We conclude that microcarrier cultures are more economical than the ordinary monolayer cultures.

Key words: Microcarriers, Cytodex 1, Cell line

INTRODUCTION

Microcarriers are a new idea in cell culture techniques pioneered by van Wezel [7].

In microcarrier cell culture, cells proliferate as a monolayer on small positively charged beads of sephadex which are suspended in a medium contained in culture bottles [3].

The large surface to volume ratio offered by the microcarrier system results in high yields of anchorage dependent cells (often as high as $5x10^6$ cells/ml with 3-5 mg microcarriers/ml) [1].

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More than 80 different cell types have been reported to grow successfuly on Cytodex 1 microcarriers [2].

Cytodex 1 microcarriers are based on a cross-linked dextran matrix which is substituted with positively charged N,N-diethyl aminoethyl (DEAE) groups to a degree which is optimal for cell growth. The charged groups are found throughout the entire matrix of the microcarrier (Fig.1).

In this study we tried to grow a rabbit kidney cell line on Cytodex 1. This cell line grew perfectly on it.

Also we established the microcarrier culture method in our laboratory. So we can try to grow other kinds of cells, particularly transformed ones by this method.

MATERIALS AND METHODS

I. CELLS AND MEDIA

The rabbit kidney cell line was obtained from World Health Organization (WHO) Geneva, Switzerland.

RK cells were grown in Eagle's minimal essential medium (MEM) supplemented with 10 % newborn calf serum and antibiotics.

II. PREPARING CYTODEX 1 FOR CULTURE

Cytodex 1 was obtained from Pharmacia Fine Chemicals, Upsala, Sweden. The dry Cytodex 1 microcarriers (1 mg/ml) were added to a glass bottle and were swollen in Ca^{++}, Mg^{++} free PBS (50-100 ml/gr Cytodex) for at least 3 hours at room temparature with occasional agitation [6]. The supernatant was decanted and the microcarriers were washed once with gentle agitation for a few minutes in fresh Ca^{++}, Mg^{++} free PBS.

After swelling the microcarriers in Ca^{++}, Mg^{++} free PBS, they were allowed to settle, the supernatant being decanted and replaced by 70 % (v/v) ethanol in distilled water.

The microcarriers were washed with this ethanol solution and then incubated overnight in 70 % (v/v) ethanol (50-100 ml/gr Cytodex) for sterilization. The ethanol solution was remowed and the sterilized microcarriers rinsed three times in sterile Ca^{++}, Mg^{++} free PBS and once in culture medium before use. Sterilized microcarriers were resuspended in a small volume of culture medium and transferred to the glass petri dish.

III. INITIATING A MICROCARRIER CULTURE

Rabbit kidney cells were put on the petri dish containing Cytodex 1 and 30 ml MEM supplemented with 10 % serum was added to the culture. Microcarrier culture was incubated at 37°C with occasional agitation.

IV. HARVESTING CELLS AND SUBCULTERING

The medium was drained from the culture and the microcarriers washed for 5 minutes in a Ca^{++}, Mg^{++} free PBS solution containing 0.02 % (w/v) EDTA,pH 7.6. The amount of EDTA PBS solution should be 50-100 ml/gr Cytodex. The EDTA PBS was remowed and replaced by trypsin-EDTA at 37°C with occasional agitation.
After 15 minutes the action of trypsin was stopped by the addition of a culture medium containing 10 % (v/v) serum.
The products of the harvesting steps were then transferred to a test tube. After 5 minutes the microcarriers settle to the bottom of the tube and the cells can then be collected in the supernatant.

RESULTS

Rabbit kidney cells attached themselves to the microcarriers 3 hours after the cells were put on Cytodex 1. RK cells grew well 10 hours after starting the microcarrier culture.

Cells grown on beads of Cytodex 1 maintained excellent growth kinetics over long periods (Fig.2). We changed the culture medium with MEM containing 3 % serum a week after the microcarrier culture started.

Culture grown on Cytodex 1 were more homogeneous than monolayer and harvesting was achieved without centrifuging the medium.

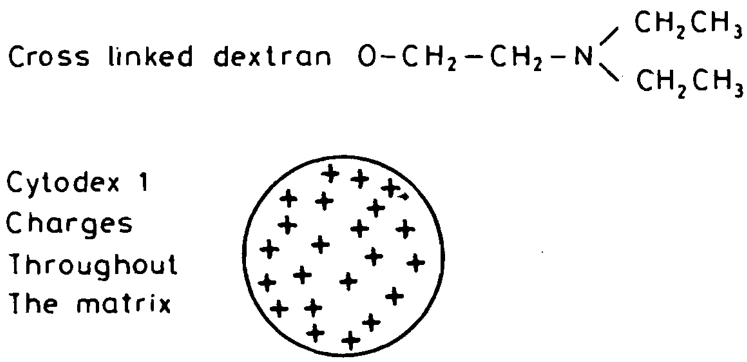

FIGURE 1. Schematic Representation of the Charged Groups on the Cytodex 1 Microcarrier

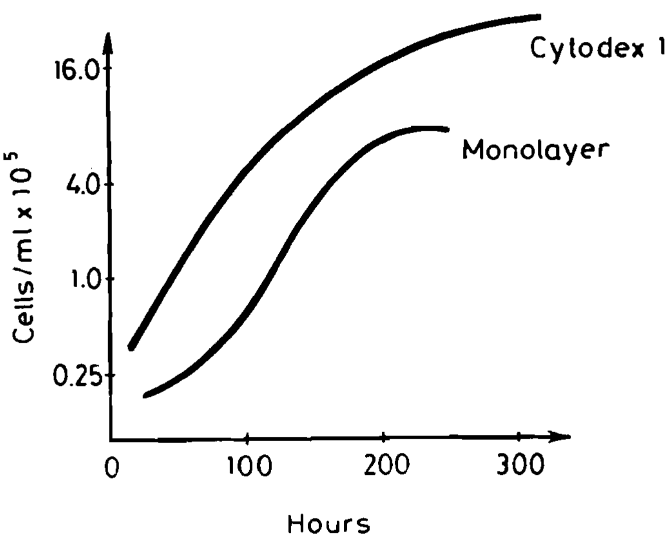

FIGURE 2. The Growth of RK Cells on Cytodex 1 Microcarriers and in Glass Bottles as Monolayer

DISCUSSION

We applied a microcarrier culture method to rabbit kindey cell line. Cells easily adapted to cytodex beads.
Cytodex beads provide a large and smooth surface for the cells to attach on [5] .

We modified some of the stages of the microcarrier culture method. We did not stir the suspension culture [4] , because the magnetic stir bar can cause collision of the beads. Collision of beads was harmful for cells. Also stirring causes the detaching of mitotic cells from the beads. We therefore put 30 ml of medium in petri dishes containing the microcarrier culture of RK cells. The occasional agitation on the petri dishes was enough to keep this kind of suspension culture healthy.
The optimal concentration of Cytodex 1 beads was 1 mg/ml for RK cells. When the concentration of Cytodex 1 was high, they precipitated at the bottom of the petri dish, and they stuck to each other. But when the Cytodex 1 concentration was decreased to 1 mg/ml, an ideal culture condition was obtained.
The microcarrier culture method we used in this study is more economic than the monolayer culture, since we used a low serum concentration in maintaining the culture medium, e.g. 3 % serum supplemented MEM was used in the microcarrier culture, while 5 % serum supplemented MEM was used in the monolayer culture.
Also RK cells in the microcarrier culture were subcultured only every 15 days, whereas RK cells in the monolayer culture were subcultured every 7 days.
We can say that Cytodex 1 saves up to 50 % of our labour because it is no longer necessary to process large numbers of petri dishes or culture bottles.

We plan to try to grow some of the poorly growing transformed cells on microcarriers.

ÖZET

Tavşan böbreği daimi hücre kültürüne mikrotaşıyıcılar (Cytodex 1) uygulandı. Tavşan böbreği daimi hücre kültürüne optimal üremesi için mikrotaşıyıcı konsantrasyonunun 1 mg/ml Cytodex 1 olduğu saptandı.

Kültür şişelerinde monolayer olarak üretilen RK hücreleri her 7 günde bir pasaj yapıldığı halde, mikrotaşıyıcılarda üretilen RK hücreleri 15 günde bir pasaj yapıldı.

Cytodex bilyeler hücreler için geniş yüzey sağladığından Cytodex 1 üzerinde üreyen RK hücreleri uzun zaman periyodunda sağlıklı üreme kinetiği gösterdiler.

Mikrotaşıyıcı kültürlerin monolayer kültürlerden daha ekonomik olduğu söylenebilir.

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SERIES B

MATHEMATICS AND STATISTICS

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MODULAR REPRESENTATIONS OF PSL(2,7) IN CHARACTERISTICS 3 and 7.*

M.I. Khanfar[(1)]

In [6] , the ordinary representations of the unimodular group G = PSL(2,7) of dimensions 3,6,7 and 8 were explicitly constructed over the complex field. The aim of this work is to investigate the modular representations of G over finite fields. This paper determines the irreducible modular representations of G in characteristics 3 and 7.

Key words: Modular representation, Characteristic, Decomposition matrix, Blocks.

1980 Subject Classification: 20C20

1. INTRODUCTION

Most of modular representation theory is due to R. Brauer. His results were stated in the language of modular characters in ([1] , [2]).

1.1. DEFINITION. Let G be a finite group and p a rational prime. An element g in G is p - regular if its order is relatively prime to p, and p - singular if its order is a power of p.

Since all conjagate elements in G are of the same order, we speak of the p - regular conjugacy classes of G.

The following two established results ([3],[7]) will be applied without further reference.

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* This work was supported by a grant from Yarmouk University.

(1) The number of absolutely irreducible modular representations of G in a modular field of characteristic p is equal to the number of p - regular conjugacy classes of G.

(2) Let Φ be an absolutely irreducible ordinary representation of G. If p^m is the highest power of p dividing the order of G and the degree of Φ, then Φ remains absolutely irreducible as a modular representation of G in characteristic p.

The following proposition is needed and can be applied in any characteristic $p \neq 2$.

1.2. PROPOSITION. Let K be an algebraically closed field of characteristic $p \neq 2$. Then $G = PSL(2,P)$ has no faithful representation of degree 2 in K [7] .

Proof. Let $f : G \longrightarrow GL(2,K)$ be a monomorphism. Then $\det : f(G) \longrightarrow K^*$ is a homomorphism onto a finite subgroup of K^*. All such subgroups of K^* are cyclic.

If $p > 3$, then $f(G)$, being simple, is a subgroup of $SL(2,K)$.
If $p = 3$, then $G \cong A_4$ and so at least $\det(f(G)) \leq \{1,-1\}$.

Choose an involution x in G such that $f(x)$ is in $SL(2,K)$. Putting $f(x)$ in rational canonical form, we find that $f(x)$ is similar to

$$\begin{bmatrix} -1 & \cdot \\ \cdot & -1 \end{bmatrix}, \text{ and therefore } f(x) = \begin{bmatrix} -1 & \cdot \\ \cdot & -1 \end{bmatrix}.$$ But then this implies that $f(x)$ is in the center of the simple group $f(G)$; a contradiction.

2. REPRESENTATIONS IN BLOCKS

We consider a method of distributing the representations of a finite group G into blocks ([3] , [4] , [5]). Let K be an algebraic number field which is a splitting field for G, and R the ring of algebraic integers in K. Let S be a prime ideal in R containing the unique rational prime p. Let R_S be the ring of S - integral elements in K. Then R_S is a principal ideal ring with quotient field K; and $\bar{K}=R/_{R_S} = R/_S$ is a modular field of characteristic p ([1] , [3] , [4]). Theory of integral representations of finite groups asserts that, every irreducible representation of G over K can be written as an integral representation of G over R_S.

Let $\tilde{C}_i$ be the sum of the elements in the conjugacy class C_i of G, i = 1, ... , n. Let Φ_i be all irreducible integral representations of G, and x_i the character of Φ_i. The sums $\tilde{C}_i$ form a K-basis for the center of the group ring KG as well as a $\bar{K}$-basis for the center of $\bar{K}G$. Thus each $\tilde{C}_k$ commutes with each g in G, and therefore the matrix $\Phi_i(\tilde{C}_k)$ commutes with the matrix $\Phi_i(g)$ for each i.Schur's Lemma asserts that each $\Phi_i(\tilde{C}_k)$ is a scalar matrix; that is

$$(*) \quad \Phi_i(\tilde{C}_k) = f_i(\tilde{C}_k)\, I, \quad 1 \leq k \leq n .$$

Since Φ_i is an integral representation, we have each $f_i(\tilde{C}_k)$ is in R_S. Taking traces in (*), we have

$$f_i(\tilde{C}_k) = \frac{|C_k|\, x_i\ (g_k)}{x_i\ (1)} \ ; \ g \text{ in } C_k, \quad 1 \leq i \ , \quad k \leq n .$$

Extending f_i to a map on the center of KG by linearity, we have

$$f_i(\tilde{C}_j\tilde{C}_k) = f_i(\tilde{C}_j)\, f_i(\tilde{C}_k) .$$

We define $\bar{f}_i$: center $(\bar{K}G) \longrightarrow \bar{K}$ by

$$f_i(\tilde{C}_k) = \overline{f_i(\tilde{C}_k)} \ , \quad i \leq k \leq n \ ;$$

where the bar denotes reduction modulo S. Two irreducible representations Φ_i and Φ_j of G belong to the same block iff $\bar{f}_i = \bar{f}_j$. If Φ_i belongs to a block B, then all irreducible modular constituents of Φ_i belong necessarily to B.

2.1. DEFINITION. Let p be a rational prime, n a rational integer. We write $v_p(n) = e$ if p^e divides n and p^{e+1} does not divide n. Assume $v_p(|G|) = m$.

2.2. DEFINITION. The defect d of a p - block B is given by

$$d = m - \min\{v_p(x_i(1)) : \Phi_i \text{ in } B\}.$$

Clearly each $d \geq 0$. A p - block of defect o contains only one irreducible representation Φ_i whose degree is divisible by p^m [1] .

3. 7-MODULAR REPRESENTATIONS OF PSL(2,7)

The simple group $G = PSL(2,7) = \langle \alpha, \beta \mid \alpha^7 = \beta^2 = (\alpha\beta)^3 = (\alpha^4\beta)^4 = 1 \rangle$ of order $168 = 2^3 \cdot 3 \cdot 7$ has 6 conjugacy classes of elements; and therefore 6 ordinary irreducible characters given below.

3.1. TABLE. Irreducible Characters.

$\lvert g \rvert$	1	2	4	3	7^+	7^-
$\lvert C(g) \rvert$	$2^3.3.7$	2^3	2^2	3	7	7
x_1	1	1	1	1	1	1
x_2	7	-1	-1	1	0	0
x_3	6	2	0	0	-1	-1
x_4	8	0	0	-1	1	1
x_5	3	-1	1	0	z	$\bar{z}$
x_6	3	-1	1	0	$\bar{z}$	z

where $|g|$ = order of $g \in G$; $|C(g)|$ = order of centralizer of g in G ; $z = \frac{-1+i\sqrt{7}}{2}$.

G has four 7 - regular conjugacy classes and hence four absolutly irreducible 7 - modular representations. The ordinary irreducible representations of degrees 1 and 7 remain irreducible as modular representations of G of characteristic 7.

By Proposition 1.2 the two ordinary irreducible representations of degree 3 are irreducible as modular representations of G in characteristic 7; but of course these representations are equivalent in this characteristic.

To determine the fourth irreducible 7 - modular representation of G, we determine first the blocks of the ordinary irreducible representations of G and the defects of these blocks. To distribute these representations into 7 - blocks, we form the table:

$$f_i(\tilde{C}_k) = \begin{bmatrix} 1 & 21 & 42 & 56 & 24 & 24 \\ 1 & 7 & . & . & -4 & -4 \\ 1 & . & . & -7 & 3 & 3 \\ 1 & -7 & 14 & . & 4\delta & 4\delta \\ 1 & -7 & 14 & . & 4\delta & 4\delta \\ 1 & -3 & -6 & 8 & . & . \end{bmatrix}$$

$$\delta = -1 + i\sqrt{7}.$$

Reducing the table modulo a prime ideal containing 7, we obtain:

$$\begin{bmatrix} 1 & . & . & . & 3 & 3 \\ 1 & . & . & . & 3 & 3 \\ 1 & . & . & . & 3 & 3 \\ 1 & . & . & . & 3 & 3 \\ 1 & . & . & . & 3 & 3 \\ 1 & -3 & 1 & 1 & . & . \end{bmatrix}.$$

Thus there are two 7-blocks:

$B_1 = \{1,3,3,6,8,\}$ of defect 1, and

$B_2 = \{7\}$ of defect 0;

where the representations in each block are indicated by their degrees. If n is the degree of the unkown irreducible 7 - modular representation of G, then 1,3, n are in B_i. The possible values for n are 4,5,6,8 . Since B_1 is of defect 1, each entry in the decomposition matrix D_1 associated with B_1 is either 0 or 1([2],[3]) It follows that the only possible value for n is 5 and D_1 has the form:

mod. deg.		1	5	3
ord. deg.	1	1	.	.
	6	1	1	.
	8	.	1	1
	3	.	.	1
	3	.	.	1

Hence the decomposition matrix D of G is of the form:

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}, \quad D_2 = [1] .$$

In fact the ordinary 6 - dimensional representation of G obtained in [6] , when reduced in characteristic 7, fixes the hyperplane $<e_j - e_1>$. If we restrict to this hyperplane,

then with respect to the basis $\{e_j - e_1 , j = 2, ... 6 \}$

we obtain:

$$\alpha \longrightarrow \begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} ; \beta \longrightarrow \begin{bmatrix} -1 & -1 & -1 & -1 & -1 \\ . & . & . & . & 1 \\ . & . & 1 & . & . \\ . & . & . & 1 & . \\ . & 1 & . & . & . \end{bmatrix} .$$

4. 3- MODULAR REPRESENTATIONS OF PSL(2,7)

G = PSL(2,7) has five 3 - regular conjugacy classes of elements, and hence five irreducible 3 - modular representations. The ordinary irreducible representations of G of degrees 1,3,3,6 remain absolutely irreducible as modular representations of G in characteristic 3.

Following the methods of the preceding section, we find that the ordinary irreducible representations of G are distributed into four 3 - blocks:

$$B_1 = \{1,7,8\} \text{ of defect } 1,$$

and $$B_2 = \{3\},\ B_3 = \{3\},\ B_4 = \{6\} \text{ of defect } 0;$$

where the representations in blocks are indicated by their degrees.

Let the unknown irreducible modular representation of G be of degree n. Then n is in B_1. The possible values for n are then 4,5, 7,8. Since B_1 is of defect 1, each entry in the decomposition matrix D_1 associated with B_1 is either 0 or 1. It follows that the only possible value for n is 7; and thus D_1 is constructed as follows:

mod. deg.		1	7
ord. deg.	1	1	.
	7	.	1
	8	1	1

Hence the decomposition matrix D of G is of the form:

$$D = \begin{bmatrix} D_1 & & & 0 \\ & D_2 & & \\ & & D_3 & \\ 0 & & & D_4 \end{bmatrix}$$

where $D_i = [1]$, i=2,3,4.

ÖZET

Karmaşık sayılar cismi üzerinde boyutları 3,6,7 ve 8 olmak üzere bir G=PSL(2,7) unimodular grubunun adi temsilleri yapılmıştır [6]. Bu çalışmanın amacı, sonlu cisimler üzerindeki G grubunun modüler temsillerini araştırmaktır. Bu araştırma 3 ve 7 karakteristikleri için G nin indirgenemeyen temsillerini belirler.

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ON THE CLASS OF PARANORMAL OPERATORS

M.Kutkut [1]

In this article we study some properties of the class of paranormal operators on an infinite dimensional separable complex Hilbert space H. We prove the following.

1. The tensor product and the direct sum of two paranormal operators are paranormal.

2. The set of paranormal operators is strongly (uniformly) closed and arcwise connected.

3. Every paranormal weighted shift is hyponormal.

4. If T is a paranormal weighted shift on H then $P_n(T)$ is paranormal on H(but $P_n(T)$ may not be hyponormal), for any polynomial P_n.

Key words: Paranormal operators, weighted shift, Hyponormal operator.

1980 Subject Classification: 47B20

1. INTRODUCTION

We consider an infinite dimensional separable complex Hilbert space H. We denote by L(H), all bounded linear operators on H. As in [1] , an operator $T \in L(H)$ is said to be **paranormal** if $||Tx||^2 \leq ||T^2x||$, for all unit vectors $x \in H$, (or equivalently $||Tx||^2 \leq ||T^2x|| \,.\, ||x||$, for every $x \in H$). Recall that an operator $T \in L(H)$ is said to be hyponormal, if $||Tx|| \geq ||T^*x||$, for every $x \in H$ (or equivalently $T^*T \geq T T^*$). An operator $T \in L(H)$ is called **normaloid**

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if

$$||T||= \sup \{|(Tx,x)| : x \in H, ||x|| = 1\} = w(T),$$

where $w(T)$ = is the numerical radius of T. Istratescu, Saito and Yoshino [5] studied some properties of paranormal operators. They proved that every hyponormal operator is paranormal, and every paranormal operator is normaloid. In [5] , it is also proved that the inverse of an invertible paranormal operator is paranormal.

In 1980, Chourasia and Ramanujan [1] studied more properties of the class of paranormal operators on Banach spaces rather than Hilbert spaces. In [1] , it is proved that every isometry is paranormal.

In this article we study further properties of paranormal operators. Let P(H) denote the class of paranormal operators on H. We prove that P(H) is arcwise connected and strongly (hence uniformly) closed. It is proved that the tensor product and the direct sum of two paranormal operators are paranormal operators. An operator $T \in L(H)$ is said to be a **weighted shift** if there is a sequence (α_n) of complex numbers and an orthonormal sequence (e_n) in H, such that $T e_n=\alpha_n e_{n+1}$; if n is an integer, T is called a **bilateral weighted** shift and if n is restricted to the positive integers, T is called a **unilateral weighted** shift; the sequence (α_n) is called the **sequence of weights.**

We should remark that there is no loss of generality in assuming that the sequence of weights (α_n) consists of positive real numbers, since two weighted shifts with weight sequences (α_n),(β_n) (resp.) are unitarily equivalent if, and only if, $|\alpha_n| = |\beta_n|$ for every integer n, (see Shields [7]), so in what follows the sequence of weights is assumed to be of positive real numbers. In [8] , it is proved that every weighted shift with a non-decreasing sequence of weights is a hyponormal operator. Thus every weighted shift with non-decreasing sequence of weights is paranormal. We prove that every paranormal weighted shift is hyponormal. We give an example of a paranormal operator which is not hyponormal. Shields [7] asked whether $P_n(T)$ is a hyponormal operator for a hyponormal unilateral weighted shift T and every polynomial P_n. In 1984, Peng Fan [6] gave a negative answer to Shield's question; he constructed a hyponormal unilateral weighted shift T for which $P_n(T)$ is not hyponormal for a given polynomial P_n. Here we prove that $P_n(T)$ is a paranormal operator for a paranormal weighted shift T and any polynomial P_n. In fact $\Phi(T)$ is a paranormal operator for a paranormal weighted shift T and any function Φ, which is analytic on the spectrum $\sigma(T)$ of T.

2. RESULTS

To be precise let $H_1 \otimes H_2$ denote the completion of the tensor product of the two Hilbert spaces H_1 and H_2. If $T \varepsilon L(H_1)$, $S \varepsilon L(H_2)$ then the tensor product $T \otimes S$ of T and S belongs to $L(H_1 \otimes H_2)$. Concerning the tensor product we prove the following.

2.1. PROPOSITION. If T,S are paranormal operators on H_1, H_2(resp), then $T \otimes S$ is paranormal.

Proof. If x,y are unit vectors, then $x \otimes y$ is a unit vector and we have

$$||T \otimes S\, x \otimes y\,||^2 = ||Tx||^2 .\, ||Sy||^2 \leq ||T^2x||.||S^2y||$$
$$\leq ||T^2 \otimes S^2\, x \otimes y|| = ||(T \otimes S)^2\, x \otimes y||,$$

which implies the conclusion of the proposition.

2.2. PROPOSITION. If $T \varepsilon L(H_1)$, $S \varepsilon L(H_2)$ are paranormal, then the direct sum $T \oplus S \varepsilon L(H_1 \oplus H_2)$ is paranormal.

Proof. If $x = x_1 \oplus x_2$ is a unit vector in $H_1 \oplus H_2$ then,

$$||T \oplus Sx||^2 = ||T\, x_1||^2 + ||S\, x_2||^2$$
$$\leq ||T^2 x_1||.||x_1|| + ||S^2 x_2||\, .||\, x_2\,|| \leq$$

$$\leq (||T^2x_1||+||S^2x_2||)\ ||x||$$
$$\leq ||T^2 \otimes S^2x||=||(T \otimes S)^2x|| ,$$

which shows that $T \otimes S$ is paranormal.

2.3. THEOREM. The set $P(H)$ is strongly (hence uniformly) closed and arcwise connected.

Proof. Let $(T_n) \subset P(H)$ be a sequence such that (T_n) converges strongly to $T \in L(H)$. Then $||T_nx-Tx|| \to 0$ as $n \to \infty$, for any vector $x \in H$, and in particular for $||x||= 1$,

$$||T x||\leq \varepsilon+||T_nx||\leq \varepsilon + ||T_n^2x||^{\frac{1}{2}},$$

for n large enough.

Since a product of operators is sequentially continuous (see Halmos [3] , problem 93 page 57), in the strong operator topology T_n^2 converges strongly to T^2. Thus,

$$||Tx||\leq \varepsilon + ||T_n^2x||^{\frac{1}{2}}$$
$$\leq \varepsilon + (\varepsilon+||T^2x||)^{\frac{1}{2}} ,$$

and since ε is arbitrary, $||Tx||^2 \leq ||T^2x||$, which means that $T \in P(H)$; $P(H)$ is strongly closed. Since every uniformly convergent sequence is strongly convergent one concludes also that $P(H)$ is uniformly closed.

To show that $P(H)$ is arcwise connected, it is enough to show that $\lambda . T \in P(H)$, for every scalar λ and $T \in P(H)$. Now let $||x||=1$, and $T \in P(H)$, then

$$||\lambda Tx||^2 = \lambda \bar{\lambda}||Tx||^2 < \lambda\bar{\lambda}||T^2x||$$
$$\leq(\lambda^2\bar{\lambda}^2||T^2x||^2)^{\frac{1}{2}}$$
$$\leq ||(\lambda T)^2x||.$$

This completes the proof of the theorem. The following proposition shows that every operator unitarily equivalent to a paranormal operator is paranormal.

2.4. PROPOSITION. Let $T \in P(H)$, then for any unitary operator u on H, $uTu^* \in P(H)$.

Proof. If $T \in P(H)$, u is a unitary operator on H, then for a unit vector $x \in H$, we have $x = uy$ for some unit vector $y \in H$, and

$$\begin{aligned} ||uTu^*x||^2 &= ||Tu^*x||^2 = ||Ty||^2 \\ &\leq ||T^2y|| = ||T^2u^*x|| \\ &\leq ||uT^2u^*x|| \leq ||(uTu^*)^2x||, \end{aligned}$$

which is as desired.

2.5. PROPOSITION. If $T, S \in P(H)$, S is an isometry which commutes with T, then $TS \in P(H)$.

Proof. If x is a unit vector in H, S is an isometry, then $y=Sx$ is a unit vector and one obtains

$$\begin{aligned} ||TSx||^2 &= ||Ty||^2 \leq ||T^2y|| \\ &\leq ||T^2Sx|| = ||ST^2Sx|| \\ &\leq ||(TS)^2x||, \end{aligned}$$

since S commutes with T, i.e., TS is paranormal.

The following proposition is concerned with the integral powers of a paranormal operator.

2.6. PROPOSITION. Let $T \in P(H)$, then for every positive integer n, $T^n \in P(H)$.

Proof. If T is paranormal then it is normaloid (see [5]) which is equivalent to saying $||T^nx|| = ||Tx||^n$, for every positive integer n. Since $||Tx||^2 \leq ||T^2x||$ for any unit vector $x \in H$, one concludes that

$$\begin{aligned} ||T^nx||^2 &= (||Tx||^2)^n \leq ||T^2x||^n \\ &\leq ||T^{2n}x|| = ||(T^n)^2x||, \end{aligned}$$

which means that T^n is paranormal.

3. WEIGHTED SHIFTS

The following example shows that the class of hyponormal operators is a proper subclass of the class of paranormal operators.

3.1. EXAMPLE. Define the weighted shift T on H, using the orthonormal sequence (e_n) of H, by the equality

$$Te_n = \begin{cases} e_{n+1}, & n \leq 2, \\ 2e_{n+1}, & n \geq 3. \end{cases}$$

This weighted shift has the following properties :

1. The operator T is hyponormal, since the sequence of weights is nondecreasing.
2. In particular T is paranormal.
3. Let $P(z) = z + az^2$, $0 < a < \frac{\sqrt{5}}{5}$, then $P(T)=T+aT^2$ is not hyponormal; for the proof (see [6]).
4. We prove that P(T) is paranormal. Indeed, elementary computation shows that:

$$P(T)e_n = \begin{cases} e_{n+1} + a\, e_{n+2}, & n \leq 1, \\ e_{n+1} + 2a\, e_{n+2}, & n = 2, \\ 2\, e_{n+1} + 4a\, e_{n+2}, & n \geq 3. \end{cases}$$

This implies that

$$||P(T)e_n||^2 = \begin{cases} 1 + a^2 & , \ n \leq 1, \\ 1 + 4a^2 & , \ n = 2, \\ 4 + 16a^2 & , \ n \geq 3. \end{cases}$$

The operator $P^2(T)$ is given by,

$$P^2(T)e_n = \begin{cases} e_{n+2}+2a\, e_{n+3} + a^2 e_{n+4}, & n \leq -1, \\ e_{n+2}+2a\, e_{n+3} + 2a^2 e_{n+4}, & n = 0, \\ e_{n+2}+4a\, e_{n+3} + 4a^2\, e_{n+4}, & n = 1, \end{cases}$$

$$P^2(T)e_n = \begin{cases} 2e_{n+2} + 8a\,e_{n+3} + 8a^2\,e_{n+4}, & n = 2, \\ 4e_{n+2} + 16ae_{n+3} + 16a^2e_{n+4}, & n \geq 3. \end{cases}$$

From which we conclude,

$$||P^2(T)e_n||^2 = \begin{cases} 1 + 4a^2 + a^4 & , n \leq -1, \\ 1 + 4a^2 + 4a^4 & , n = 0, \\ 1 + 16a^2 + 16a^4 & , n = 1, \\ 4 + 64a^2 + 64a^4 & , n = 2, \\ 16 + (16)^2a^2 + (16)^2a^4, & n \geq 3. \end{cases}$$

By comparing the value of $||P(T)e_n||^4$ and $||P^2(T)e_n||^2$ one concludes that $||P(T)e_n||^2 \leq ||P^2(T)e_n||$, which is as desired.

3.2. REMARK. The restriction on a to be such that $0 < a < \frac{\sqrt{5}}{5}$ is needed in [6] to show that P(T) is not hyponormal but it is not needed to show that P(T) is paranormal.

3.3. EXAMPLE. Let T be the weighted shift defined on (e_n) by

$$T(e_n) = \begin{cases} \frac{1}{2}\,e_{n+1} & , n < 0 \\ e_{n+1} & , n \geq 0. \end{cases}$$

It is clear that T is hyponormal and therefore it is paranormal. Hartman [4] showed that the spectrum $\sigma(T)$ of T is not a spectral set of T. Recall that a subset X of the complex plane is said to be a spectral set of T if $\sigma(T) \subset X$ and for any rational function Φ with poles off X we have $||\Phi(T)|| < \sup_{Z \in X} |\Phi(Z)|$ $(= ||\Phi||_\infty)$.

This example shows also that it is not necessary for the spectrum of a paranormal operator to be a spectral set.

The following proposition shows that every paranormal weighted shift must be hyponormal.

3.4. PROPOSITION. If T is a paranormal weighted shift, then its sequence of weights (α_n) satisfies the inequality

$$\alpha_n^2 \le \alpha_n \cdot \alpha_{n+1}.$$

Proof. If (e_n) is the orthonormal sequence on which T is defined, then $\|Te_n\|^2 = \alpha_n^2$, and $\|T^2 e_n\| = \alpha_n \cdot \alpha_{n+1}$ and since T is paranormal then $\alpha_n^2 \le \alpha_n \cdot \alpha_{n+1}$.

Example 3.1(4) is not only true for $P(z) = z + az^2$ but it is true for any polynomial. This is shown in the following.

3.5. THEOREM. Let $T \in P(H)$ be a weighted shift, Let $P(z)$ be a polynomial of degree k. Then $P(T)$ is a paranormal operator.

Proof. Let (e_n) be an orthonormal sequence in H, (α_n) the weight sequence. If T defined by $Te_n = \alpha_n e_{n+1}$ is paranormal then $\alpha_n \le \alpha_n \cdot \alpha_{n+1}$ for every integer n. If $P(z)$ is a polynomial of degree k, i.e., $P(z) = 1 + a_1 z + a_2 z^2 + \ldots + a_k z^k$, a_i complex numbers, $i = 1, 2, \ldots, k$, then an elementary computation shows that,

$$\|P(T)e_n\|^2 = 1 + |a_1|^2 \alpha_n^2 + |a_2|^2 \alpha_{n+1}^2 + \ldots + |a_k|^2 \alpha_n^2 \cdot \alpha_{n+1}^2 \cdots \alpha_{n+k}^2$$

and

$$\|P^2(T)e_n\|^2 = 1 + 4|a_1|^2 \alpha_n^2 + |2a_2 + a_1 \bar{a}_1|^2 \alpha_n^2 \cdot \alpha_{n+1}^2 + \ldots + |a_k|^2 \alpha_n^2 \cdot \alpha_{n+1}^2 \cdots \alpha_{n+2k}^2.$$

By comparing $\|P(T)e_n\|^4$ and $\|p^2(T)e_n\|^2$ one concludes that $\|P(T)e_n\|^2 \le \|P^2(T)e_n\|$, for every integer n, which shows that $P(T)$ is paranormal as desired.

3.6. THEOREM. Let $T \in P(H)$ be a weighted shift. If Φ is any analytic function on the spectrum $\sigma(T)$ of T, then $\Phi(T) \in P(H)$.

Proof. Since Φ is an analytic function then there is a sequence of polynomials (P_k) which converges uniformly to Φ. By Theorem 3.5 $P_k(T)$ is paranormal for every k. By Theorem 2.3 the uniform limit $\Phi(T) = $ u-lim $P_k(T)$ is paranormal.

We conclude this article with the following

3.7. THEOREM. If $T, S \in P(H)$ are weighted shifts then $T + S$, $TS \in P(H)$.

Proof. If (α_n), (β_n) are the weight sequences of T, S resp. and (e_n) the orthonormal sequence in H, on which both are defined, then $\alpha_n^2 \leq \alpha_n \cdot \alpha_{n+1}$ and $\beta_n^2 \leq \beta_n \cdot \beta_{n+1}$. For the product TS, we have $||TSe_n||^2 = \alpha_{n+1}^2 \cdot \beta_n^2$ while, $||(TS)^2 e_n|| = \alpha_{n+1} \cdot \alpha_{n+3} \cdot \beta_n \cdot \beta_{n+2}$, for every integer n, therefore $||TSe_n||^2 \leq ||(TS)^2 e_n||$. Thus TS is paranormal. For the sum $T + S$, *direct calculation shows that*

$$||(T+S)e_n||^2 = \alpha_n^2 + 2\alpha_n \beta_n + \beta_n^2,$$

and

$$||(T+S)^2 e_n||^2 = \alpha_n^2 \cdot \alpha_{n+1}^2 + 2\alpha_n \cdot \alpha_{n+1}^2 \cdot \beta_n + 2\alpha_n^2 \cdot \alpha_{n+1} \beta_{n+1} +$$

$$+ 4\alpha_n \cdot \alpha_{n+1} \beta_n \cdot \beta_{n+1} + \beta_n^2 \cdot \alpha_{n+1}^2 + 2\beta_n^2 \cdot \beta_{n+1} \cdot \alpha_{n+1} +$$

$$+ \alpha_n^2 \cdot \beta_{n+1}^2 + 2\alpha_n \cdot \beta_n \beta_{n+1}^2 + \beta_n^2 \cdot \beta_{n+1}^2 .$$

Comparing the values of $||(T+S)e_n||^4$ and $||(T+S)^2 e_n||^2$, one concludes that

$$||(T+S)e_n||^2 \leq ||(T+S)^2 e_n||;$$

for every integer n, which is the required conclusion.

ÖZET

Bu çalışmada sonsuz boyutlu ayrılabilir karmaşık bir H Hilbert uzayı üzerinde paranormal operatörlerin sınıfının bazı özellikleri üzerinde çalışıyoruz. Aşağıdakileri kanıtlıyoruz.

1. İki paranormal operatörlerin tensör çarpımı ve direkt toplamı paranormaldir.
2. Paranormal operatörlerin kümesi kuvvetli (düzgün) kapalı ve yay bağlantılıdır.
3. Her paranormal yüklenmiş kayma hiponormaldir.

4. T,H üzerinde paranormal yüklenmiş kayma ise bu durumda, her P_n polinomu için $P_n(T)$,H üzerinde paranormaldir (ama hiponormal olmayabilir).

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ERGODICITY OF HILBERT SPACE OPERATORS

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In this article the following theorem is proved. Theorem: If T is an ergodic operator (in the uniform, strong or weak operator topology) on an infinite dimensional complex Hilbert space and F is a continuous multiplicative function then F(T) is ergodic (in the respective topology). This result implies that STS^{-1} is ergodic for any invertible operator S : the adjoint T^* of T is ergodic. If T is subnormal then the minimal normal extension N of T is ergodic. Moreover the dual S of a pure subnormal operator T is ergodic.

Key words: Hilbert space, Multiplicative function, Ergodic operator

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1. INTRODUCTION

Let X be a Banach space, and L(X) the algebra of all bounded linear operators on X. Lotz in [3] introduced the following definition of ergodicity.

1.1. DEFINITION. Let $G \subset L(X)$ be a multiplicative semi-group of operators. Denote by **chG** the convex hull of G. Then G is said to be **ergodic** if the closure of chG has a zero element, i.e., there exists a projection operator P in the closure of chG such that PS = SP=P for every operator S in the closure of chG. If the closure of ch G is taken in the uniform operator topology then G is called **uniformly ergodic** and if the closure of chG is taken in the strong operator

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operator topology then G is called **strongly ergodic.** If the underlying space is an inner product space then **weakly ergodic** may be similarly defined. Denote these closures of chG by uchG, schG and wchG respectively.

1.2. DEFINITION. Let $T\varepsilon L(X)$, then T is said to be ergodic (in the uniform, strong, or weak topology (for inner product space)) if the cyclic semigroup $G_T=\{T^n: n=0,1,2, \ldots\}$ is ergodic in the respective topology.

2. RESULTS

In this paper we study ergodicity of operators on Hilbert space. If H is an infinite dimensional complex Hilbert space, L(H) denotes the algebra of all bounded linear operators on H.

Let U be the unitary group on H. The unitary orbit U(T) of $T\varepsilon L(H)$ is defined by

$$U(T)=\{u\,T\,u^* \ : u\,\varepsilon\, U,\ u^* = \text{adjoint of } u\}.$$

The similarity orbit S(T) of $T\varepsilon L(H)$ is defined by

$$S(T)=\{s\,T\,s^{-1}: s \text{ is invertible}\}.$$

Now, we are ready to introduce our results.

2.1. LEMMA. If F is a 1-1 multiplicative function, and $T\varepsilon L(H)$, such that $F(T)\varepsilon L(H)$, then F induces a multiplicative function F between G_T and $G_{F(T)}$, which is one to-one and onto.

Proof. Define $F : G_T \longrightarrow G_{F(T)}$, by

$$F(T^n) := F^n(T).$$

It is an easy matter to show that F is multiplicative, one-to-one and onto.

2.2. REMARK. $F \neq 0, I, P$; P is a projection.

2.3. LEMMA. The multiplicative function F defined in Lemma 2.1 is extendable to a multiplicative function denoted by F (for simplicity) between chG_T and $ch\ G_{F(T)}$, which is also one-to-one and onto.

Proof. If $A \in ch\ G_T$ then there exist positive integers $n_1, \ldots, n_k$ and non-negative real numbers $a_1, \ldots, a_k : \Sigma_i a_i = 1$ and $A = \Sigma_i a_i T^{n_i}$. Now the extension of F is defined by $F(A) = \Sigma_i a_i F(T^{n_i}) = \Sigma_i a_i F^{n_i}(T)$. If $B \in ch\ G_T$ then $B = \sum_j b_j T^{m_j}$, for some positive integers $m_1, \ldots, m_\ell$ and some non-negative real numbers $b_1, \ldots, b_l$ whose sum is one, and thus $AB = \sum_i \sum_j a_i b_j T^{n_i + m_j}$, and since $\Sigma_{i,j} a_i b_j = 1$, $AB \in chG_T$ that is, chG_T is a semi-group (See Lotz [3], p. 146). The function F is multiplicative, since

$$\begin{aligned} F(AB) &= \Sigma_{i,j}\ a_i b_j\ F(T^{n_i + m_j}) \\ &= \Sigma_{i,j}\ a_i b_j\ F(T^{n_i})\ F(T^{m_j}) \\ &= \Sigma_i a_i\ F(T^{n_i}))\ (\Sigma_j b_j\ F(T^{m_j})) \\ &= F(A).F(B). \end{aligned}$$

It is not difficult to show that F is one-to-one and onto.

2.4. LEMMA. If the multiplicative function F is continuous, then the function F defined in Lemma 2.3 is extendable to a multiplicative function (denoted by F) between the closure of $ch\ G_T$ and the closure of $ch\ G_{F(T)}$, which is also one-to-one, onto and continuous.

Proof: In the uniform topology, if $A \in uch\ G_T$, then there is $(A_i) \subset ch\ G_T$ such that $||A - A_i|| \longrightarrow 0$ as $i \longrightarrow \infty$. Since F is continuous and multiplicative $chG_{F(T)}$ and $F(chG_T)$ can be identified. By continuity of F, $F(A_i)$ converges uniformly to $F(A)$ and we can define $F: uch\ G_T \longrightarrow uch\ G_{F(T)}$ by $F(A) = u\text{-lim. } F(A_i)$, where u-lim. means uniform limit. From the definition of F, it is clear that F is continuous on $uchG_T$.

Since F is one-to-one in Lemma 2.3 then by linearity, F is one-to-one on uch G_T. If $B \varepsilon$ uch $G_{F(T)}$, then there is a sequence $(B_i) \subset$ ch $G_{F(T)}$ such that (B_i) converges uniformly to B. Since $B_i \varepsilon$ ch $G_{F(T)}$, then there is $A_i \varepsilon$ ch G_T such that $B_i = F(A_i)$ (because F is onto by lemma 2.3). Since F is continuous and one-to-one, it is invertible and thus (A_i) converges uniformly to $A \varepsilon$ uch G_T, so that by the definition of F, F(A) = B or F is onto.

If $A,B \varepsilon$ uch G_T then A=u-lim. A_i, B=u-lim B_i for $(A_i),(B_i) \subset chG_T$.

Since multiplication is continuous (and in particular sequentially continuous) in the uniform topology (see Halmos [2] , problem 91, page 57) we have, AB=u-lim A_iB_i. This implies that:

$$F(A\,B) = \text{u-lim } F(A_iB_i) = \text{u-lim } F(A_i).\, F(B_i)$$
$$= \text{u-lim } F(A_i)\text{u-lim } F(B_i) = F(A)\,F(B),$$

since F is multiplieative on ch G_T by Lemma 2.3. Thus F is multiplicative on uch G_T.

For the strong topology, a similar argument can be given, since the product is sequentially continuous in the strong operator topology (see Halmos [2] , problem 93 page 57).

Since the product is not even sequentially continuous in the weak operator topology, (see Halmos[2], problem 93 page 57) we provide the following argument to show that the extension F:wch $G_T \longrightarrow$ wch $G_{F(T)}$ is multiplicative.

It is known that if $A_i \longrightarrow A$ weakly then $A_iB \longrightarrow AB$, $BA_i \longrightarrow BA$ weakly for any fixed operator B(see Halmos [2] problem 92 page 57).
If $A \varepsilon$ wch G_T and $B \varepsilon$ ch G_T, then there is (A_i) in ch G_T such that $(A_i) \longrightarrow A$, weakly and thus $(A_iB) \longrightarrow AB$ weakly, as $i \longrightarrow \infty$; and by the continuity of F, (by the definition of F in the weak operator topology), and since F is multiplicative on ch G_T,

$$F(A\,B) = \text{w-lim } F(A_iB) = \text{w-lim } F(A_i)\,F(B)$$
$$= F(A).\,F(B), \qquad \ldots \quad (2.1)$$

where w-lim means weak-limit.

Now, assume that both A, $B \in$ wch G_T, then there exist (B_i) in ch G_T: B=w-lim B_i. Therefore $AB_i \longrightarrow AB$ weakly as $i \longrightarrow \infty$ and $AB \in$ ch G_T, this implies that (by continuity of F in the weak topology):

$$\begin{aligned} F(AB) &= \text{w-lim } F(A.B_i) \\ &= \text{w-lim } F(A).F(B_i), \text{by } (2.1) \\ &= F(A).F(B) , \end{aligned}$$

i.e., F is multiplicative on wch G_T.

2.5. THEOREM. Let $T \in L(H)$ be an ergodic (in the uniform, strong or weak operator topology) operator, then for a 1-1 continuous multiplicative function F, for which $F(T) \in L(H)$, F(T) is ergodic.

Proof. By Lemma 2.4, F induces a continuous multiplicative function denoted by F, (for simplicity) which is one-to-one and onto between the closure of ch G_T and the closure of ch$G_{F(T)}$, (in the uniform, strong, or weak operator topology). If P is a zero element of the closure of ch G_T then F(P) is a zero element of the closure of ch $G_{F(T)}$, (in the respective topology). Indeed, since P is a zero element, then PS = SP = P, for every S in the closure of ch G_T. If A is an element in the closure of ch $G_{F(T)}$ there is S in the closure of ch G_T such that A= F(S), since F is one-to-one and onto. Since F is also multiplicative one obtains,

$$\begin{aligned} F(P).A &= F(P).F(S) = F(P.S) = F(P) \\ &= F(S.P) = F(S).\ F(P) \\ &= A.F(P). \end{aligned}$$

This implies that F(T) is ergodic.

We prove the following results as corollaries of the Theorem.

We should remark that the term ergodic is understood in the three topologies unless otherwise mentioned.

2.6. COROLLARY. Let $T \in L(H)$ be ergodic. If S is an invertible operator on H, then $ST\,S^{-1}$ is ergodic.

Proof. Define $F:G_T \longrightarrow G_{STS^{-1}}$ by $F(T^n)= S\, T^n S^{-1}$. It is clear that F is 1-1, multiplicative and continuous (for any fixed operator S, see Halmos [2] problem 92). It is also one-to-one and onto. By the Theorem $F(T) = S\, T\, S^{-1}$ is ergodic.

2.7. REMARK. Corollary 2.6. means that if T is ergodic then every element in the similarity orbit S(T) of T is ergodic.

2.8. REMARK. Since every unitary operator u is invertible and $u^{-1}=u^*$, the adjoint of u, it follows from Corollary 2.6, that $u\, T\, u^*$ is ergodic and thus, every element of the unitary orbit U(T) is ergodic.

2.9. COROLLARY. If $T \varepsilon L(H)$ is ergodic then the adjoint T^* of T is ergodic.

Proof. Define $F: G_T \longrightarrow G_{T^*}$ by $F(T^n)={T^*}^n$ which is a multiplicative isometry (one-to-one) and onto. Moreover, F is continuous (in the uniform and weak topology but not in the strong, see Halmos [2] problem 90 page 56). By the Theorem $F(T) = T^*$ is ergodic (in the uniform and weak operator topologies).

Since, uch $G_{T^*} \subset$ sch $G_{T^*} \subset$ wch G_{T^*}, one concludes that schG_{T^*} has a zero element, i.e. T^* is strongly ergodic.

2.10. COROLLARY. If $T \varepsilon L(H)$ is ergodic, then F(T) is ergodic, for every analytic multiplicative function F, on the spectrum $\sigma(T)$ of T.

Proof. It is not difficult to show that a multiplication analytic function is one-to-one, onto, and continuous. (see Remark 2.11). By the Theorem, F(T) is ergodic.

2.11. REMARK. If F(z) is analytic, where z is a complex variable, then

$$F(z)= \sum_0^\infty a_n z^n \text{ and } F(z)\, F(z) = F(z^2), \text{ thus}$$

$$\left(\sum_0^\infty a_n z^n\right)\left(\sum_0^\infty a_n z^n\right) = \sum_0^\infty a_n z^{2n}$$

and this is true if $a_n = \begin{cases} 1 & \text{for some } n \\ 0 & \text{otherwise} \end{cases}$ and thus $F(z)=z^n$.

2.12. REMARK. Corollary 2.10 and the preceding remark imply that if T is ergodic then T^n is ergodic for every positive integer n and thus every element in the cyclic semi-group G_T is ergodic.

For a subnormal operator T, let N be the minimal normal extension of T (which is unique up to unitary equivalence, see Halmos [2], problem 155, page 101). Conway [1] showed that N can be written as a two-by-two matrix with operator entries.

$$N = \begin{bmatrix} T & A \\ 0 & S^* \end{bmatrix},$$

where $N \in L(K)$, N is normal and K is a Hilbert space such that $K = H \oplus H^1$. If the decomposition $K = H^1 \oplus H$ is considered then the adjoint N^* of N is given by

$$N^* = \begin{bmatrix} S & A^* \\ 0 & T^* \end{bmatrix}$$

The operator T is said to be pure subnormal if T is subnormal and neither A nor T is normal. Olin([4] Lemma 5.3) has observed that T is pure subnormal if, and only if, N^* is the minimal normal extension of S, and S is called the dual of T.

2.13. COROLLARY. Let $T \in L(H)$ be an ergodic subnormal operator. If N is the (unique) minimal extension of T, then N is ergodic.

Proof. If f is an analytic function, then it is proved in Conway [1] that $f(N)$ is the minimal normal extension of $f(T)$. Using the decomposition mentioned above,

$$F(N) = \begin{bmatrix} f(T) & Y \\ 0 & f(S^*) \end{bmatrix},$$

where N is defined on $H \oplus H^1$.

In particular let f be multiplicative and analytic and so, by remark 2.11, $f(z) = z^n$, for some n. Define the multiplicative continuous function $F: G_T \to G_N$ by $F(T^n) = N^n$, by the preceding argument F is the minimal normal extension of T^n, F is one-to-one and onto. By the theorem, N=F(T) is ergodic.

2.14. REMARK. Since N is unique, by symmetry the proof of Corollary 2.13 implies that if N is ergodic then T is also.

Finally, we arrive at the following result concerning the dual of a pure subnormal operator.

2.15. COROLLARY. If $T \in L(H)$ is a pure subnormal operator, and if S is the dual of T, then T is ergodic if and only if S is ergodic.

Proof. Let N be the minimal normal extension of T, then (by Olin's result [4]) the adjoint N^* is the minimal normal extension of the dual S of T if, and only if, T is pure. Thus if T is ergodic then by corollary 2.13, N is also ergodic. By Corollary 2.9 N^* is ergodic, and by Remark 2.14, S is ergodic. By the symmetry of the proof we have if S is ergodic then T is ergodic too.

ÖZET

Bu çalışmada aşağıdaki teorem kanıtlanmıştır.

TEOREM. T, bir sonsuz boyutlu karmaşık Hilbert uzayı üzerinde (düzgün, kuvvetli veya zayıf operatör topolojisine göre), bir ergodik operatör ve F çarpımsal sürekli bir fonksiyon ise aynı topolojiye göre F(T) ergodiktir. Böylece herhangi bir terslenebilir S operatörü için STS^{-1} ergodiktir; T nin T^*eki ergodiktir. T alt normal ise T nin minimal normal genişlemesi N ergodiktir. Üstelik, bir pür alt normal T operatörünün S duali ergodiktir.

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ON THE COEFFICIENTS OF CERTAIN MEROMORPHIC FUNTIONS

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The aim of this work is to obtain the sharp bounds for the coefficients of the functions belonging to the class of meromorphic functions which are analytic in $0<|z|<1$.

Key words: Analytic function, Meromorphic function, Starlike function of order α.

1980 Subject Classification: 30D30

1. INTRODUCTION

An analytic function

$$g(z)=\frac{1}{z}+b_1z+b_2z^2+\dots$$

is said to be starlike of order α, $(0\leq\alpha<1)$ in the punctured disc $K=\{z:0<|z|<1\}$ if and only if

$$\mathrm{Re}\left\{\frac{-zg'(z)}{g(z)}\right\}>\alpha$$

for all z in the unit disc $E=\{z:|z|<1\}$.

Let F_u be the class of functions

$$f(z)=\frac{1}{z}+a_1z+a_2z^2+\dots$$

which are analytic in K and satify the condition

$$\left|\frac{zf'(z)}{g(z)}+1\right|<\left|u\frac{zf'(z)}{g(z)}-1\right|,\ (0\leq u\leq 1),z\in E \qquad \dots(1.1)$$

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where

$$g(z)=\frac{1}{z}+b_1z+b_2z^2+\dots$$

is analytic and starlike of order α in K.

Owa [3] has obtained some coefficient relations for the class F_u taking

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

analytic in the unit disc E,

$$g(z)=z-\sum_{n=2}^{\infty}b_nz^n \quad , \; (b_n\geq 0)$$

analytic and starlike of order α in E.

A special subclass of F_u was studied by Kaczmarkski [1].

Pommerenke [4] has obtained the relation

$$|b_n|\leq \frac{2(1-\alpha)}{n+1} \quad , \quad n=1,2,\dots \qquad \dots\dots\dots\dots (1.2)$$

for the meromorphic function

$$g(z)=\frac{1}{z}+b_1z+b_2z^2+\dots$$

which is starlike of order α in K.

In this paper, using this result we obtain the coefficients relation for the class F_u.

2. RESULT

2.1. THEOREM. If $f(z)\varepsilon F_u$ and Re $a_k\bar{b}_k\geq 0$ for $k=1,2,3,\dots(n-2)$ then

$$n\,|a_n|\leq 1+u+\frac{2(1-\alpha)}{n+1} \quad ,n\geq 1.$$

The bounds are sharp.

Proof: Since $f(z)\varepsilon F_u$ we obtain

$$\frac{zf'(z)}{g(z)}+1=w(z)\left[u\,\frac{Zf'(z)}{g(z)}-1\right] \quad \dots (2.1)$$

where $w(z)=C_2z^2+C_3z^3+\dots$ and $|w(z)|<1$ in E. On substituting the the power series for f(z), g(z) and w(z) in (2.1) we have

$$\sum_{k=1}^{\infty} (ka_k+b_k)z^k = \left[-\frac{1+u}{z} + \sum_{k=1}^{\infty} (u\,k\,a_k-b_k)z^k \right] w(z)$$

$$\sum_{k=1}^{\infty} (ka_k+b_k)z^{k+1} = \left[-(1+u) + \sum_{k=1}^{\infty} (u\,k\,a_k-b_k)z^{k+1} \right] w(z) \quad \ldots(2.2)$$

Equating coefficients of z^2 and z^3 on both sides of (2.2) we get

$$a_1+b_1 = -(1+u)C_2,$$

and

$$2a_2+b_2 = -(1+u)C_3.$$

Using $|C_2|\leq 1$, $|C_3|\leq 1$ and from (1.2) we obtain

$$|a_1|\leq|a_1+b_1| + |b_1| \leq 1+u+1-\alpha \quad \ldots(2.3)$$

and

$$2|a_2|\leq|2a_2+b_2| + |b_2|\leq 1+u+\frac{2(1-\alpha)}{3}. \quad \ldots(2.4)$$

Equating the coefficients of z^n $(n>2)$ on both sides of (2.2) we get

$$na_n+b_n = -(1+u)C_{n+1}+(ua_1-b_1)C_{n-1}+$$

$$\ldots + [u(n-2)a_{n-2}-b_{n-2}]C_2. \quad \ldots(2.5)$$

From (2.2) and (2.5) we obtain

$$\sum_{k=1}^{n} (ka_k+b_k)z^{k+1} + \sum_{k=n+1}^{\infty} (ka_k+b_k)z^{k+1} =$$

$$= [-(1+u)+\sum_{k=1}^{n-2} (u\,k\,a_k-b_k)z^{k+1}]\, w(z) + \sum_{k=n+2}^{\infty} d_k z^k \ldots(2.6)$$

Since $\sum_{k=n+1}^{\infty} (ka_k+b_k)z^{k+1} = \sum_{k=n+2}^{\infty} [(k-1)a_{k-1}-b_{k-1}]\, z^k$

and from (2.6) we have

$$\sum_{k=1}^{n} (ka_k+b_k)z^{k+1} + \sum_{k=n+2}^{\infty} e_k z^k =$$

$$\left[-(1+u)+\sum_{k=1}^{n-2} (u\,k\,a_k-b_k)z^{k+1} \right] w(z). \quad \ldots(2.7).$$

Using $|w(z)|<1$ in E and Parseval identity([2],p:100) on both sides of (2.7) we obtain

$$\sum_{k=1}^{n} |ka_k+b_k|^2 \cdot r^{2(k+1)} + \sum_{k=n+2}^{\infty} |e_k|^2 \cdot r^{2k} < (1+u)^2 +$$

$$+ \sum_{k=1}^{n-2} | u\, k\, a_k - b_k |^2 \cdot r^{2(k+1)} \,. \qquad \ldots (2.8)$$

If we let $r \longrightarrow 1$, from (2.8) we have

$$\sum_{k=1}^{n} |ka_k+b_k|^2 \leq (1+u)^2 + \sum_{k=1}^{n-2} | u\, k\, a_k - b_k |^2 \qquad \ldots (2.9)$$

or

$$|na_n+b_n|^2 \leq (1+u)^2 - \sum_{k=1}^{n-2} (1-u^2)k^2 |a_k| - \sum_{k=1}^{n-2} 2k(1+u)\mathrm{Re}\, a_k \bar{b}_k$$

$$- |(n-1)a_{n-1}+b_{n-1}|^2 \,. \qquad \ldots (2.10)$$

Since $o\leq u\leq 1$ and $\mathrm{Re}\, a_k\bar{b}_k \geq 0$ $(k=1,2,\ldots(n-2)$ it follows that

$$| na_n+b_n | \leq 1+u \,. \qquad \ldots (2.11)$$

From (1.2) and (2.11) we have

$$n|a_n| \leq |na_n+b_n| + |b_n| \leq 1+u+ \frac{2(1-\alpha)}{n+1} \,, \; n=3,4,5,\ldots \qquad \ldots (2.12)$$

Hence from (2.3), (2.4) and (2.12) we obtain

$$n|a_n| \leq 1+u+ \frac{2(1-\alpha)}{n+1} \text{ for } n=1,2,3,\ldots$$

Now let us show that the bounds are sharp.

We take

$$\frac{zf'(z)}{g(z)} = \frac{-(1+z^{n+1})}{1-uz^{n+1}}$$

and

$$g(z) = \frac{1}{z} (1+z^{n+1})^{\frac{2(1-\alpha)}{n+1}} \,.$$

Since $\mathrm{Re}\, \{ \frac{zf'(z)}{g(z)} \} > \alpha$, $g(z)$ is meromorphic starlike of order α in K and

$$\left[\frac{\frac{zf'(z)}{g(z)}+1}{\frac{Uzf'(z)}{g(z)}-1}\right] = |z^{n+1}|<1.$$

It follows that $f(z)\varepsilon F_u$.

On the other hand ,

$$zf'(z)=\frac{-(1+z^{n+1})}{1-uz^{n+1}} \cdot \frac{1}{z}(1+z^{n+1})^{\frac{2(1-\alpha)}{n+1}}$$

has the expansion

$$zf'(z)=-\frac{1}{z}-\left(1+u+\frac{2(1-\alpha)}{n+1}\right)z^n-\ \ldots$$

and we have

$$n|a_n|= 1+u+\frac{2(1-\alpha)}{n+1} .$$

Hence the proof is complete.

ÖZET

$g(z)=\frac{1}{z}+b_1z+b_2z^2+\ \ldots$ fonksiyonu $K=\{\ z:0<|\ z\ |<1\ \}$ kümesinde α-mertebeden yıldızıl olmak üzere, $E=\{\ z:|z|<1\}$ de

$$\left|\ \frac{zf'(z)}{g(z)}+1\ \right|<\left|u.\frac{zf'(z)}{g(z)}-1\right| \quad (0\leq u\leq 1)$$

koşulunu gerçekleyen ve K da analitik olan

$$f(z)=\frac{1}{z}+a_1z+a_2z^2+\ \ldots$$

fonksiyonlarının ailesini F_u ile gösterelim. Bu çalışmada F_u ailesine ait olan f(z) fonksiyonlarının katsayıları ile ilgili

$$n|a_n|\leq 1+u+\frac{2(1-\alpha)}{n+1}$$

biçiminde kesin sınırın varlığı gösterilmiştir.

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A CHARACTERIZATION OF UNITS IN ZS_4

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In this work we characterize the units in the integral group ring ZS_4 by using their images in certain general linear groups under the distinct inequivalent irreducible representations of the group S_4. The group of units in ZS_4 of augmentation 1 is shown to be isomorphic with a certain subgroup of $GL(2,Z) \oplus GL(3,Z) \oplus GL(3,Z)$.

Key words: Group, Ring, Representation, Unit, Character
1980 Subject Classification: 16A26

1. INTRODUCTION

Let $U(ZG)$ denote the group of units of the integral group ring ZG of a group G over the ring Z of integers. Hughes and Pearson [4]and Allen and Hobby[1] gave characterizations of $U(ZS_3)$ and $U(ZA_4)$, respectively. Milies [3] did the same for $U(ZD_4)$. Dennis [6] and Sehgal [5] pointed to the need for additional work with some small groups, including determination of the units of the rational group ring QG. In this article we restrict ourselves to integral group rings and obtain a characterization of $U(ZS_4)$, where S_4 is the symmetric group of degree 4.
Let $V(ZG)$ denote the units $\Sigma a_i g_i$ in ZG which have coefficient sum $\Sigma a_i=1$. The technique used by Hughes and Pearson consists of making use of the distinct irreducible inequivalent representations of S_3 to obtain a 6x6 matrix P that describes a faithful representation.

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$$\theta : V(ZS_3) \longrightarrow \theta(V(ZS_3)) \subset GL(2,Z).$$

When $r=\Sigma a_i g_i \in ZS_3$, the entries of $\theta(r)$ are obtained from the matrix product $\alpha P=\beta$, where $\alpha=[a_1 a_2 \ldots a_6]$ is the row-matrix of coefficients of r. Finally, by computing the inverse matrix P^{-1} and solving the linear system of congruences obtained by requiring that $\alpha=\beta P^{-1}$ have entries in Z, they obtained necessary and sufficient conditions that describe the matrices in GL(2,Z) which belong to $\theta(V(ZS_3))$.

2. RESULT

Just following this method, we use the inequivalent irreducible representations of the group S_4 to find conditions determining the elements of $U(ZS_4)$. The group S_4 can be generated by the cycles a=(12) and b=(234) which are subject to the relations

$$a^2=b^3=(1) \qquad \text{and} \qquad ab^2=(ba)^3.$$

We agree always to list the elements of S_4 in accordance with the conjugate classes in the following order:

$$[(1)=g_1] \quad \begin{bmatrix} (12)=a & =g_2 \\ (13)=bab^2 & =g_3 \\ (14)=b^2ab & =g_4 \\ (23)=abab^2a & =g_5 \\ (24)=ab^2aba & =g_6 \\ (34)=ab^2abab^2 & =g_7 \end{bmatrix} \quad \begin{bmatrix} (123)=bab^2a & =g_8 \\ (124)=b^2aba & =g_9 \\ (134)=aba & =g_{10} \\ (234)=b & =g_{11} \\ (132)=abab^2 & =g_{12} \\ (142)=ab^2ab & =g_{13} \\ (143)=ab^2a & =g_{14} \\ (243)=b^2 & =g_{15} \end{bmatrix}$$

$$\begin{bmatrix} (1234)=ab & =g_{16} \\ (1243)=ab^2 & =g_{17} \\ (1324)=bab & =g_{18} \\ (1432)=b^2a & =g_{19} \\ (1342)=ba & =g_{20} \\ (1423)=b^2ab^2 & =g_{21} \end{bmatrix} \quad \begin{bmatrix} (12)(34)=(bab)^2 & =g_{22} \\ (13)(24)=(ab)^2 & =g_{23} \\ (14)(23)=(ab^2)^2 & =g_{24} \end{bmatrix}$$

The group S_4 has five inequivalent irreducible representations

$\rho_1, \rho_2, \rho_3, \rho_4$ and ρ_5 of degrees 1,1,2,3 and 3, respectively; ρ_1 being the 1-representation and ρ_2 the representation assigning to each cycle its sign. For ease of computation we choose the representations ρ_3, ρ_4, and ρ_5 slightly different from those arising naturally from Young diagrams:

$\rho_1(a) = 1$ $\qquad$ $\rho_1(b) = 1$

$\rho_2(a) = -1$ $\qquad$ $\rho_2(b) = 1$

$$\rho_3(a) = A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \rho_3(b) = B = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

$$\rho_4(a) = C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \rho_4(b) = D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\rho_5(a) = -C = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad \rho_5(b) = D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Let $\rho_1 \oplus \rho_2 \oplus \rho_3 \oplus \rho_4 \oplus \rho_5$ denote the direct sum of the irreducible representations of S_4. When $g \in S_4$, $\rho(g) = X^*$ is a 10x10 matrix with blocks on the main diagonal as follows:

$$X^* = \begin{bmatrix} x_1 & & & & & & & & & \\ & x_2 & & & & & & & O & \\ & & x_3 & x_4 & & & & & & \\ & & x_5 & x_6 & & & & & & \\ & & & & x_7 & x_8 & x_9 & & & \\ & & & & x_{10} & x_{11} & x_{12} & & & \\ & & & & x_{13} & x_{14} & x_{15} & & & \\ & & & & & & & x_{16} & x_{17} & x_{18} \\ & O & & & & & & x_{19} & x_{20} & x_{21} \\ & & & & & & & x_{22} & x_{23} & x_{24} \end{bmatrix} = \begin{bmatrix} K & & & O \\ & X_1 & & \\ & & X_2 & \\ O & & & X_3 \end{bmatrix} \quad ..(2.1)$$

We use K, X_1, X_2 and X_3 to denote, respectively, the diagonal matrix with x_1, x_2 on the main diagonal; the 2x2 matrix whose entries are x_3, x_4, x_5, x_6; the 3x3 matrix whose entries are $x_7, x_8, \ldots,$

x_{15} and the 3x3 matrix whose entries are $x_{16}, x_{17}, \ldots, x_{24}$.

The products in S_4 will be computed by cycling from the right; e.g. $(12)(134)=(1342)$. Select the elements $g_i \in S_4$ in the given order and let x_i denote the 24-dimensional row-vector corresponding to $\rho(g_i)$. The components of x_i constitute the ith row of the following 24x24 matrix:

$$P=\begin{bmatrix}
1&1&1&0&0&1&1&0&0&0&1&0&0&0&1&1&0&0&0&1&0&0&0&1\\
1&-1&0&1&1&0&0&0&1&0&-1&0&1&0&0&0&0&-1&0&1&0&-1&0&0\\
1&-1&1&-1&0&-1&-1&0&0&0&0&1&0&1&0&1&0&0&0&0&-1&0&-1&0\\
1&-1&-1&0&-1&1&0&1&0&1&0&0&0&0&-1&0&-1&0&-1&0&0&0&0&1\\
1&-1&-1&0&-1&1&0&-1&0&-1&0&0&0&0&-1&0&1&0&1&0&0&0&0&1\\
1&-1&1&-1&0&-1&-1&0&0&0&0&-1&0&-1&0&1&0&0&0&0&1&0&1&0\\
1&-1&0&1&1&0&0&0&-1&0&-1&0&-1&0&0&0&0&1&0&1&0&1&0&0\\
1&1&-1&1&-1&0&0&0&-1&1&0&0&0&-1&0&0&0&-1&1&0&0&0&-1&0\\
1&1&0&-1&1&-1&0&-1&0&0&0&1&-1&0&0&0&-1&0&0&0&1&-1&0&0\\
1&1&-1&1&-1&0&0&0&1&-1&0&0&0&-1&0&0&0&1&-1&0&0&0&-1&0\\
1&1&0&-1&1&-1&0&1&0&0&0&1&1&0&0&0&1&0&0&0&1&1&0&0\\
1&1&0&-1&1&-1&0&1&0&0&0&-1&-1&0&0&0&1&0&0&0&-1&-1&0&0\\
1&1&1&1&-1&0&0&0&-1&-1&0&0&0&1&0&0&0&-1&-1&0&0&0&1&0\\
1&1&0&-1&1&-1&0&-1&0&0&0&-1&1&0&0&0&-1&0&0&0&-1&1&0&0\\
1&1&-1&1&-1&0&0&0&1&1&0&0&0&1&0&0&0&1&1&0&0&0&1&0\\
1&-1&1&-1&0&-1&1&0&0&0&0&-1&0&1&0&-1&0&0&0&0&1&0&-1&0\\
1&-1&-1&0&-1&1&0&1&0&-1&0&0&0&0&1&0&-1&0&1&0&0&0&0&-1\\
1&-1&0&1&1&0&0&0&-1&0&1&0&1&0&0&0&0&1&0&-1&0&-1&0&0\\
1&-1&1&-1&0&-1&1&0&0&0&0&1&0&-1&0&-1&0&0&0&0&-1&0&1&0\\
1&-1&-1&0&-1&1&0&-1&0&1&0&0&0&0&1&0&1&0&-1&0&0&0&0&-1\\
1&-1&0&1&1&0&0&0&1&0&1&0&-1&0&0&0&0&-1&0&-1&0&1&0&0\\
1&1&1&0&0&1&-1&0&0&0&1&0&0&0&-1&-1&0&0&0&1&0&0&0&-1\\
1&1&1&0&0&1&1&0&0&0&-1&0&0&0&-1&1&0&0&0&-1&0&0&0&-1\\
1&1&1&0&0&1&-1&0&0&0&-1&0&0&0&1&-1&0&0&0&-1&0&0&0&1
\end{bmatrix}
\begin{array}{l}
(1)\\ (12)\\ (13)\\ (14)\\ (23)\\ (24)\\ (34)\\ (123)\\ (124)\\ (134)\\ (234)\\ (132)\\ (142)\\ (143)\\ (243)\\ (1234)\\ (1243)\\ (1324)\\ (1432)\\ (1342)\\ (1423)\\ (12)(34)\\ (13)(24)\\ (14)(23)
\end{array}
\quad (2.2)$$

The direct sum representation ρ can be extended by linearity to a Z-algebra homomorphism $\bar{\rho}$ from ZS_4 into a Z-algebra of 10x10 matrices of the form given in (2.1). Let $p(X^*)=(p_1(X^*), p_2(X^*), p_3(X^*))=(X_1, X_2, X_3)$ be the natural projection mapping X^* to the matrices X_1, X_2, X_3. The composite map obtained by applying $\bar{\rho}$ and then p is a Z-algebra homomorphism sending ZS_4 into the Z-algebra of triples of 2x2, 3x3 and 3x3 matrices. We let θ denote the restriction of this map $p \circ \bar{\rho}$ to $V(ZS_4)$. Then θ is a homomorphism of $V(ZS_4)$ into the group $GL(2,Z) \oplus GL(3,Z) \oplus GL(3,Z)$.

The homomorphism θ can be described in terms of the matrix P as follows:

Let $\alpha=[a_1 a_2 \ldots a_{24}]$ represent the element $r=\sum_{i=1}^{24} a_i g_i$, where the supporting elements g_i are listed as mentioned before. It follows that the matrix product $\alpha P=x^*$ gives the row-vector x^* associated with $\bar{\rho}(r)=X^*$. Then $\theta(r)=P(x^*)$. The image of $V(ZS_4)$ under θ consists of the elements (X_1,X_2,X_3) of $GL(2,Z) \oplus GL(3,Z) \oplus GL(3,Z)$ which are projections of those matrices X^* such that $\alpha P=x^*$, where α is the row-vector of coefficients of some $r \in V(ZS_4)$. Thus, once P^{-1} is known, we can say that the range of θ is contained in the set of all (X_1, X_2,X_3) in $GL(2,Z) \oplus GL(3,Z) \oplus GL(3,Z)$ such that $X^* P^{-1}$ is a row-vector of integers whose sum is 1.

The matrix P can be inverted using Schur relations, as mentioned in [1,2]. We list the steps of this inversion process for the sake of completeness. (ρ_k's are the irreducible representations of S_4)

(1) Determine the fixed i,j and k such that the mth column of P consists of $\{\rho_k(g)_{ij} \mid g \in S_4\}$.

(2) Once i,j and k are known, select the column of P; say the m_tth column, which consists of $\{\rho_k(g)_{ji} \mid g \in S_4\}$.

(3) Rearrange the m_tth column by interchanging the entries for $\rho_k(g)_{ji}$ and $\rho_k(g^{-1})_{ji}$. Then multiply each entry by $n_k/24$ where n_k is the degree of ρ_k.

(4) Transpose the result of step (3) to obtain the mth row of P^{-1}.

This leads to the following matrix:

$$P^{-1}=(1/24)\begin{bmatrix}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&-1&-1&-1&-1&-1&-1&1&1&1&1&1&1&1&1&-1&-1&-1&-1&-1&-1&1&1&1\\
2&0&2&-2&-2&2&0&0&-2&0&-2&-2&0&-2&0&2&-2&0&2&-2&0&2&2&2\\
0&2&0&-2&-2&0&2&2&-2&2&-2&-2&2&-2&2&0&-2&2&0&-2&2&0&0&0\\
0&2&-2&0&0&-2&2&-2&2&-2&2&2&-2&2&-2&-2&0&2&-2&0&2&0&0&0\\
2&0&-2&2&2&-2&0&-2&0&-2&0&0&-2&0&-2&-2&2&0&-2&2&0&2&2&2\\
3&0&-3&0&0&-3&0&0&0&0&0&0&0&0&0&3&0&0&3&0&0&-3&3&-3\\
0&0&0&3&-3&0&0&0&-3&0&3&3&0&-3&0&0&3&0&0&-3&0&0&0&0\\
0&3&0&0&0&0&-3&-3&0&3&0&0&-3&0&3&0&0&-3&0&0&3&0&0&0\\
0&0&0&3&-3&0&0&3&0&-3&0&0&-3&0&3&0&-3&0&0&3&0&0&0&0\\
3&-3&0&0&0&0&-3&0&0&0&0&0&0&0&0&0&0&3&0&0&3&3&-3&-3\\
0&0&3&0&0&-3&0&0&3&0&3&-3&0&-3&0&-3&0&0&3&0&0&0&0&0\\
0&3&0&0&0&0&-3&0&-3&0&3&-3&0&3&0&0&0&3&0&0&-3&0&0&0\\
0&0&3&0&0&-3&0&-3&0&-3&0&0&3&0&3&3&0&0&-3&0&0&0&0&0\\
3&0&0&-3&-3&0&0&0&0&0&0&0&0&0&0&0&-3&0&0&3&0&-3&-3&3\\
3&0&3&0&0&3&0&0&0&0&0&0&0&0&0&-3&0&0&-3&0&0&-3&3&-3\\
0&0&0&-3&3&0&0&0&-3&0&3&3&0&-3&0&0&-3&0&0&3&0&0&0&0\\
0&-3&0&0&0&0&3&-3&0&3&0&0&-3&0&3&0&0&3&0&0&-3&0&0&0\\
0&0&0&-3&3&0&0&3&0&-3&0&0&-3&0&3&0&3&0&0&-3&0&0&0&0\\
3&3&0&0&0&0&3&0&0&0&0&0&0&0&0&0&0&-3&0&0&-3&3&-3&-3\\
0&0&-3&0&0&3&0&0&3&0&3&-3&0&-3&0&3&0&0&-3&0&0&0&0&0\\
0&-3&0&0&0&0&3&0&-3&0&3&-3&0&3&0&0&0&-3&0&0&3&0&0&0\\
0&0&-3&0&0&3&0&-3&0&-3&0&0&3&0&3&-3&0&0&3&0&0&0&0&0\\
3&0&0&3&3&0&0&0&0&0&0&0&0&0&0&0&-3&0&0&-3&0&-3&-3&3
\end{bmatrix} \qquad (2.3)$$

Since the matrix P is invertible, the Z-algebra homomorphism π is an isomorphism. The blocks of the matrix X^* in (2.1) are

$$K=\left[\begin{array}{c|c} \bar{a}_1+a_2+\ldots+a_{24} & \\ \hline & \begin{array}{l} a_1-a_2-\ldots-a_7+a_8+\ldots+a_{15} \\ -a_{16}-\ldots-a_{21}+a_{22}+a_{23}+a_{24} \end{array} \end{array}\right]$$

$$X_1=\left[\begin{array}{l|l} \begin{array}{l} a_1+a_3-a_4-a_5+a_6-a_8-a_{10}-a_{13} \\ -a_{15}+a_{16}-a_{17}+a_{19}-a_{20}+a_{22}+a_{23}+a_{24} \end{array} & \begin{array}{l} a_2-a_3-a_6+a_7+a_8-a_9+a_{10}-a_{11} \\ -a_{12}+a_{13}-a_{14}+a_{15}-a_{16}+a_{18} \\ -a_{19}+a_{21} \end{array} \\ \hdashline \begin{array}{l} a_2-a_4-a_5+a_7-a_8+a_9-a_{10}+a_{11}+a_{12} \\ -a_{13}+a_{14}-a_{15}-a_{17}+a_{18}-a_{20}+a_{21} \end{array} & \begin{array}{l} a_1-a_3+a_4+a_5-a_6-a_9-a_{11}-a_{12} \\ -a_{14}-a_{16}+a_{17}-a_{19}+a_{20}+a_{22} \\ +a_{23}+a_{24} \end{array} \end{array}\right]$$

$$X_2=\left[\begin{array}{c|c|c} a_1-a_3-a_6+a_{16} & a_4-a_5-a_9+a_{11} & a_2-a_7-a_8+a_{10} \\ +a_{19}-a_{22}+a_{23}-a_{24} & +a_{12}-a_{14}+a_{17}-a_{20} & -a_{13}+a_{15}-a_{18}+a_{21} \\ \hline a_4-a_5+a_8-a_{10} & a_1-a_2-a_7+a_{18} & a_3-a_6+a_9+a_{11} \\ -a_{13}+a_{15}-a_{17}+a_{20} & +a_{21}+a_{22}-a_{23}-a_{24} & -a_{12}-a_{14}-a_{16}+a_{19} \\ \hline a_2-a_7-a_9+a_{11} & a_3-a_6-a_8-a_{10} & a_1-a_4-a_5+a_{17} \\ -a_{12}+a_{14}+a_{18}-a_{21} & +a_{13}+a_{15}+a_{16}-a_{19} & +a_{20}-a_{22}-a_{23}+a_{24} \end{array}\right] \quad (2.4)$$

$$X_3=\left[\begin{array}{c|c|c} a_1+a_3+a_6-a_{16} & -a_4+a_5-a_9+a_{11} & -a_2+a_7-a_8+a_{10} \\ -a_{19}-a_{22}+a_{23}-a_{24} & +a_{12}-a_{14}-a_{17}+a_{20} & -a_{13}+a_{15}+a_{18}-a_{21} \\ \hline -a_4+a_5+a_8-a_{10} & a_1+a_2+a_7-a_{18} & -a_3+a_6+a_9+a_{11} \\ -a_{13}+a_{15}+a_{17}-a_{20} & -a_{21}+a_{22}-a_{23}-a_{24} & -a_{12}-a_{14}+a_{16}-a_{19} \\ \hline -a_2+a_7-a_9+a_{11} & -a_3+a_6-a_8-a_{10} & a_1+a_4+a_5-a_{17} \\ -a_{12}+a_{14}-a_{18}+a_{21} & +a_{13}+a_{15}-a_{16}+a_{19} & -a_{20}-a_{22}-a_{23}+a_{24} \end{array}\right].$$

Consider a vector $x^*=[x_1,x_2,\dots,x_{24}]$. The vector $[a_1,a_2,\dots,a_{24}]$ whose image under θ is x^* will be computed from the product x^*P^{-1}. For each of a_i to be an integer, the system $x^*P^{-1}\equiv 0 \pmod{24}$ must be solved. We list some properties of the matrices X_1, X_2 and X_3 which can be drawn from the forms of these matrices (2.4).

For $X_1=\begin{bmatrix} x_3 & x_4 \\ x_5 & x_6 \end{bmatrix}$; $X_2=\begin{bmatrix} x_7 & x_8 & x_9 \\ x_{10} & x_{11} & x_{12} \\ x_{13} & x_{14} & x_{15} \end{bmatrix}$; $X_3=\begin{bmatrix} x_{16} & x_{17} & x_{18} \\ x_{19} & x_{20} & x_{21} \\ x_{22} & x_{23} & x_{24} \end{bmatrix}$

we have

(a) in X_1: $x_3+x_5\equiv x_4+x_6 \pmod 3$

(b) in X_2: $x_7+x_{10}+x_{13}\equiv x_8+x_{11}+x_{14}\equiv x_9+x_{12}+x_{15} \pmod 2$

in X_3: $x_{16}+x_{19}+x_{22}\equiv x_{17}+x_{20}+x_{23}\equiv x_{18}+x_{21}+x_{24} \pmod 2$

(c) the corresponding entries of X_2 and X_3 are congruent (mod 2):

$x_7\equiv x_{16} \pmod 2$; $x_8\equiv x_{17} \pmod 2$; ... ; $x_{15}\equiv x_{24} \pmod 2$

To describe some other relations between the matrices X_3 and X_2 we define the following sums and products:

$$\begin{aligned} t_1 &= x_7+x_{11}+x_{15} & \bar{t}_1 &= x_7x_{11}x_{15}\\ t_2 &= -x_8+x_{12}-x_{13} & \bar{t}_2 &= x_8x_{12}x_{13}\\ t_3 &= -x_9+x_{10}-x_{14} & \bar{t}_3 &= x_9x_{10}x_{14}\\ t_4 &= x_9-x_{11}+x_{13} & \bar{t}_4 &= -x_9x_{11}x_{13}\\ t_5 &= x_8+x_{10}-x_{15} & \bar{t}_5 &= -x_8x_{10}x_{15}\\ t_6 &= -x_7+x_{12}+x_{14} & \bar{t}_6 &= -x_7x_{12}x_{14} \end{aligned}$$

$$\begin{aligned} t_1' &= x_{16}+x_{20}+x_{24} & \bar{t}_1' &= x_{16}x_{20}x_{24}\\ t_2' &= -x_{17}+x_{21}-x_{22} & \bar{t}_2' &= x_{17}x_{21}x_{22}\\ t_3' &= -x_{18}+x_{19}-x_{23} & \bar{t}_3' &= x_{18}x_{19}x_{23}\\ t_4' &= x_{18}-x_{20}+x_{22} & \bar{t}_4' &= -x_{18}x_{20}x_{22}\\ t_5' &= x_{17}+x_{19}-x_{24} & \bar{t}_5' &= -x_{17}x_{19}x_{24}\\ t_6' &= -x_{16}+x_{21}+x_{23} & \bar{t}_6' &= -x_{16}x_{21}x_{23}. \end{aligned}$$

In terms of these t_k and t_k' s we have the following relations between X_2 and X_3: $(k=1,\dots,6)$

(d) Let x_i and x_j be any two elements of X_2 belonging to t_k and let x_i' and x_i' be the corresponding elements of X_3 in t_k'. Then

(a') If x_i,x_j (resp x_i',x_j') belong to t_1,t_2 or t_3 (resp. t_i',t_2' or t_3') then

$$x_i+x_j\equiv-(x_i'+x_j') \pmod 4 \quad\text{and}\quad x_i-x_j\equiv-(x_i'-x_j') \pmod 4$$

(b') If x_i,x_j (resp. x_i',x_j') belong to t_4,t_5 or t_6 (resp. t_4',t_5' or t_6') then

$$x_i+x_j \equiv x_i'+x_j' \pmod 4 \quad\text{and}\quad x_i-x_j \equiv x_i'-x_j' \pmod 4.$$

We know that for a vector x^* to belong to $\theta(V(ZS_4))$ a necessary and sufficient condition is that the vector x^* satisfy the congruences $x^* P^{-1}\equiv 0 \pmod{24}$. This gives 24 equations represented by the matrix equality $[a_1,a_2,\dots,a_{24}]= x^* P^{-1}$ for calculating the integers $a_1,a_2,\dots,a_{24}$.

Summing up these equations we see that $\Sigma_{i=1}^{24} a_i= 24x_1/24=x_1$, from which we derive the conclusion that the first component x_1 of the vector x^* must be chosen to be 1. Because the diagonal entries of the matrix K must be units, the second component x_2 of x^* will be chosen to be ± 1. Hence the pair $(x_1;x_2)$ must be $(1,1)$ or $(1,-1)$. For these choices of (x_1,x_2) the 5^{th}, 6^{th}, 7^{th}, 10^{th}, 11^{th}, 12^{th}, 23^{rd} and 24^{th} of the congruences $x^* P^{-1} \equiv 0 (\mathrm{mod}\ 24)$ will not be affected; 1^{st}, 2^{nd}, 3^{rd}, 4^{th}, 8^{th} and 9^{th} of these congruences will have the following forms in terms of the sums t_k and t'_k.$(k=1,\ldots,6)$.

For $(x_1,x_2)=(1,1)$	For $(x_1,x_2)=(1,-1)$
$2(1+x_3+x_6)\ +3(t_1+t'_1)\equiv 0(\mathrm{mod}24)$	$2(x_3+x_6)\ +3(t_1+t'_1)\equiv 0(\mathrm{mod}24)$
$2(x_4+x_5)\ +3(t_4-t'_4)\equiv 0$ "	$2(1+x+x_5)+(t_4-t'_4)\equiv 0$ "
$2(x_3-x_5-x_6)\ +3(t_6-t'_6)\equiv 0$ "	$2(1+x_3-x_5-x_6)+3(t_6-t'_6)\equiv 0$ "
(2.5). $2(-x_3-x_4+x_6)+3(t_5-t'_5)\equiv 0$ "	$2(1-x_3-x_4+x_6)+3(t_5-t'_5)\equiv 0$ "
$2(1+x_4-x_5-x_6)+3(t_3+t'_3)\equiv 0$ "	$2(x_4-x_5-x_6)\ +3(t_3+t'_3)\equiv 0$ "
$2(1-x_3-x_4+x_5)+3(t_2+t'_2)\equiv 0$ "	$2(-x_3-x_4+x_5)+3(t_2+t'_2)\equiv 0$ ".

To derive further properties of matrices X_2 and X_3 we consider the sums t_i of X_2(similarly for X_3).
From the form of X_2 in (2.4) we write the sums:

$$t_1=3a_1-a_2-a_3-a_4-a_5-a_6-a_7+a_{16}+a_{17}+a_{18}+a_{19}+a_{20}+a_{21}-a_{22}-a_{23}-a_{24}$$
$$t_2=-a_2+a_3-a_4+a_5-a_6+a_7+3a_9-a_{11}-a_{12}-a_{14}-a_{16}-a_{17}-a_{18}+a_{19}+a_{20}+a_{21}$$
$$t_3=-a_2-a_3+a_4-a_5+a_6+a_7+3a_8-a_{10}-a_{13}-a_{15}-a_{16}-a_{17}+a_{18}+a_{19}+a_{20}-a_{21}.$$

Keeping in mind that $a_1+\ldots+a_{24}=1$, we see that $t_1+t_2+t_3+1\equiv 0(\mathrm{mod}\ 2)$ and that the sum $t_1+t_2+t_3$ is an odd integer. Similarly, the sums $t_4+t_5+t_6$, $t'_1+t'_2+t'_3$ and $t'_4+t'_5+t'_6$ are odd. Accordingly, we can list the following properties:

(1) Since $t_i\equiv t'_i$ (mod 2), either all of the sums t_1,t_2,t_3 are odd, or one is odd and the other two are even, and similarly for the sums t'_1,t'_2,t'_3 ; t_4,t_5,t_6; t'_4,t'_5,t'_6.

(2) For at least one t_i(resp.t'_i) all three numbers belonging to t_i(resp.t'_i) are odd; otherwise, all six products $\overline{t}_i$ being even, the determinant of $X_2=\overline{t}_1+\ldots+\overline{t}_6$ would be$\neq\pm 1$.

(3) For at most one t_i(resp.t'_i) all three numbers are odd. Otherwise $\det(X_2)$,(resp. $\det(X_3)$) would be even.

Accordingly, for the matrices X_2 and X_3, for exactly one t_i all **three** numbers belonging to these t_i, t_i' are odd, the remaining elements of these matrices are even.

(4) If the odd entries of X_2 and X_3 belong to t_1 and t_1' then at least one of the entries on $t_2, t_3 (t_2', t_3')$ must be divisible by 4.

The projections of $\theta(V(ZS_4))$ under p_1, p_2, p_3 are, respectively,

$$G_1 = \left\{ \begin{bmatrix} x_3 & x_4 \\ x_5 & x_6 \end{bmatrix} \middle| x_3 + x_5 \equiv x_4 + x_6 \pmod 3 \right\} \subset GL(2,Z)$$

$$G_2 = \left\{ \begin{bmatrix} x_7 & x_8 & x_9 \\ x_{10} & x_{11} & x_{12} \\ x_{13} & x_{14} & x_{15} \end{bmatrix} \middle| \begin{array}{l} \text{column sums are congruent (mod 2)} \\ \text{and exactly one } t_i \text{ contains odd} \\ \text{entries; the other entries all} \\ \text{being even.} \end{array} \right\} \subset GL(Z,3)$$

$G_3 \subset GL(3,Z)$ defined similarly for G_2.

Now let $G = G_1 \oplus G_2 \oplus G_3$

$$= \{ (X_1, X_2, X_3) \in GL(2,Z) \oplus GL(3,Z) \oplus GL(3,Z) \mid X_1, X_2, X_3 \text{ satisfy conditions (a)-(d) and (2.5)} \}.$$

Then G is a group containing $\theta(V(ZS_4))$ and we can characterize the group of units under consideration as follows:

2.1. PROPOSITION. $\theta(V(ZS_4)) \cong G$.

Proof. The only thing we need to show is that $\theta : V(ZS_4) \longrightarrow G$ is an isomorphism and that the inclusion $\theta(V(ZS_4)) \subset G$ is an equality. Let I_2 and I_3 denote identity matrices of order 2 and 3, resp. and K the diagonal matrix with entries x_1, x_2. Consider the matrix

$$X^* = \begin{bmatrix} K & & & \\ & I_2 & & \\ & & I_3 & \\ & & & I_3 \end{bmatrix}$$

If we choose (x_1, x_2) other than $(1,1)$; that is, if $(x_1, x_2) = (1,-1)$, then we get from the product $x^* P^{-1}$, entries other than integers; e.g. the first four entries of $x^* P^{-1}$ are 11/12, 1/12, 1/12 and 1/12.

Therefore, the natural projection p restricted to $\bar{\rho}(V(ZS_4))$ has trivial kernel. Since $\bar{\rho}$ is already an isomorphism, it follows that θ is an isomorphism.

Finally, we show that θ is onto by observing that $[a_1,\ldots,a_{24}] = x^*P^{-1}$ has integer entries. Choose $(X_1,X_2,X_3)\varepsilon G$. From the forms of the blocks in (2.4) we obtain, by a long computation, the congruence

$$\det(X_1)=x_3x_6-x_4x_5 \equiv x_1x_2 \pmod 3) \equiv x_2 \pmod 3)$$

from which we determine the second entry of K to be $x_2=\det(X_1)$. Now we use the choice $(x_1,x_2)=(1,\det(X_1))$ together with the entries of X_1,X_2,X_3 to determine the vector $x^*=[x_1,\ldots,x_{24}]=[1, \det X_1, x_3, \ldots, x_{24}]$ and find the vector of coefficients $\alpha=[a_1,\ldots,a_{24}]$ for an element $r=\Sigma a_i g_i$ in $V(ZS_4)$ which satisfies the equality $\alpha P=x^*$. That the coefficients a_1,a_2,a_3,a_4,a_8 and a_9 are integers follows from the conditions (2.5) and that the remaining coefficients are integers from conditions (a), (b), (c), (d).

Example. We observe that the triple

$$(X_1,X_2,X_3)= \left(\begin{bmatrix} 46 & -19 \\ -63 & 26 \end{bmatrix}, \begin{bmatrix} 4 & -30 & 33 \\ -6 & 51 & -58 \\ -3 & 22 & -24 \end{bmatrix}, \begin{bmatrix} -52 & -30 & -21 \\ -6 & -3 & -2 \\ -57 & -34 & -24 \end{bmatrix} \right)$$

belongs to the group G described above. Because $\det(X_1)=-1$ we make the choice $(x_1,x_2)=(1,-1)$ and form the vector

$x^*=[1,-1,46,-19,-63,26,4,-30,33,-6,51,-58,-3,22,-24,-52,-30,-21,$
$-6,-3,-2,-57,-34,-24]$.

Now the product x^*P^{-1} gives the vector

$\alpha=[0,0,0,0,0,0,-27,0,0,6,-30,0,0,0,0,28,0,0,0,0,0,24,0,0]$

with non-zero entries $a_7=-27, a_{10}=6, a_{11}=-30, a_{16}=28$, $a_{22}=24$

and $a_1+\ldots+a_{24}=1$. Hence the group ring element

$r=a_7g_7+a_{10}g_{10}+a_{11}g_{11}+a_{16}g_{16}+a_{22}g_{22}$
$=-27(34)+6(134)-30(234)+28(1234)+24(12)(34)\varepsilon ZS_4$

is a unit with inverse $r^{-1}\varepsilon ZS_4$ determined by $\alpha^{-1}=(x^*)^{-1}P^{-1}$; where $(x^*)^{-1}$ is a vector obtained by using 1, $\det X_1$ and the entries of the matrices X_1^{-1}, X_2^{-1} , X_3^{-1} . Since

$$(X_1^{-1},X_2^{-1},X_3^{-1}) = \left(\begin{bmatrix} -26 & -19 \\ -63 & -46 \end{bmatrix}, \begin{bmatrix} 52 & 6 & 57 \\ 30 & 3 & 34 \\ 21 & 2 & 24 \end{bmatrix}, \begin{bmatrix} -4 & 6 & 3 \\ 30 & -51 & -22 \\ -33 & 58 & 24 \end{bmatrix} \right)$$

and $\det(X_1^{-1})=-1$, we again make the choice $(X_1,X_2)=(1,-1)$ and form $(x^*)^{-1}=[1,-1,-26,-19,-63,-46,52,6,57,30,3,34,21,2,24,-4,6,3,30$
$-51,-22,-33,58,24]$.

Accordingly,

$$\alpha^{-1}=(x^*)^{-1}P^{-1} =[0,0,0,0,0,0,-27,0,0,0,0,0,0,-6,30,0,0,0,28,0,0,-24,0,0]$$

with non-zero elements $a_7=-27; a_{14}=-6; a_{15}=30; a_{19}=28; a_{22}=-24$, and the desired inverse element r^{-1} is obtained as

$$r^{-1}=a_7g_7+a_{14}g_{14}+a_{15}g_{15}+a_{19}g_{19}+a_{22}g_{22}$$
$$=-27(34)-6(143)+30(243)+28(1432)-24(12)(34).$$

ÖZET

Bu çalışmada ZS_4 grup halkasının birimselleri, S_4 grubunun farklı denk olmayan indirgenemez representasyonları altındaki muayyen genel lineer gruplar içindeki görünümleri kullanılarak karakterize edilmiş olup, ZS_4 de ogmentasyonu 1 olan birimseller grubunun $GL(2,Z) \oplus GL(3,Z) \oplus GL(3,Z)$ nin bir alt grubuna izomorf olduğu gösterilmiştir.

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A NOTE ON FUZZY NEARLY COMPACT SPACES

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This paper discusses fuzzy near compactness in fuzzy topological spaces. We give some characterizations of fuzzy near compactness in terms of regular open and regular closed fuzzy sets.

Key Words: Fuzzy topological spaces, Fuzzy near compactness

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1. INTRODUCTION

Zadeh in [8] introduced the fundamental concept of a fuzzy set. Fuzzy topological spaces were first introduced in the literature by Chang [2] , who studied several basic concepts including fuzzy continuous maps and compactness. In this paper we study fuzzy nearly compact spaces. We give some characterizations of near compactnes in terms of regular open or regular closed fuzzy sets. We first give some necessary preliminaries.

Let X be a nonempty set and $F(X)=\{f \mid f:X \longrightarrow [0,1]\}$. The elements of F(X) are called fuzzy subsets of X [8] . We denote by 0_X and 1_X the functions on X identically equal to 0 and 1 respectively.

Now we recall that a fuzzy topology in the sense of Chang [2] is a subset τ_X of I^X such that

(i) $0_X \varepsilon \tau_X$ and $1_X \varepsilon \tau_X$,

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(ii) If $f, g \in \tau_X$, then $f \wedge g \in \tau_X$,

(iii) If $f_i \in \tau_X$, $\forall i \in I$, then $\bigvee_{i \in I} f_i \in \tau_X$.

A collection $\{f_i\}_{i \in I}$, where $f_i \in \tau_X$, $i \in I$, is a cover of X iff $\bigvee_{i \in I} f_i = 1_X$. A fuzzy topological space is compact iff every open cover has a finite subcover [2] .

Let X be a fuzzy topological space. For a fuzzy set of X, the closure $\bar{f}$ and the interior $\overset{o}{f}$ of f are defined respectively as

$$\bar{f} = \inf\{g : g > f, g' \in \tau_X\}$$

and

$$\overset{o}{f} = \sup\{g : g < f, g \in \tau_X\}.$$

A fuzzy set f is called regularly open iff $f = (\bar{f})^o$ and regularly closed iff $f = (\overset{o}{f})^-$ [1] .

A fuzzy topological space X is almost compact iff every open cover has a finite subcollection whose closures cover X [3] . A fuzzy topological space X is called nearly compact iff every open cover of X has a finite subcollection such that the interiors of the closures of fuzzy sets in this collection covers X[4] .

If τ_X is a fuzzy topology on X, a collection $B \subseteq \tau_X$ is a base of τ_X iff each $f \in \tau_X$ is of the form $\bigvee_{i \in I} f_i$, where $f_i \in B, \forall i$; and its members are called the "basic open sets of the topology τ_X". A collection $S \subseteq \tau_X$ is a subbase iff $\{f_{i_1} \wedge \ldots \wedge f_{i_n}\}$ $f_i \in S$ is a base of τ_X.

2. RESULTS

The following theorem shows that we may work with fuzzy regularly closed or fuzzy regularly open sets:

2.1. THEOREM. In a fuzzy topological space X with base B the following conditions are equivalent:

(i) X is nearly compact.

(ii) Every basic fuzzy open cover of X has a finite subcollection such that the interiors of closures of fuzzy sets in this subcollection covers X.

(iii) Every cover of X by fuzzy regularly open sets has a finite subcover.

(IV) Every collection of fuzzy regularly closed sets having the finite intersection property has nonempty intersection.

(V) Every collection $\{f_i\}_{i\varepsilon I}$ of fuzzy closed sets having the property that for any finite subcollection $\{f_i: i=1,\ldots,n\}$ of $\{f_i\}_{i\varepsilon I}$, $\bigwedge_{i=1}^{n}(\overset{o}{f_i})^{-}\neq 0_X$, has nonempty intersection.

Proof. (i$\Rightarrow$ii) follows easily.

(ii$\Rightarrow$iii). Let $\{f_i\}_{i\varepsilon I}$ be any fuzzy regularly open cover of X and let B be a base for τ_X. For each $i\varepsilon I$,

$f_i= V\{g_j:j\varepsilon I_i, g_j\varepsilon B\}$. Then $A=\{g_j: j\varepsilon I_i, i\varepsilon I\}$ is a basic open cover of X. By(ii), A has a finite subcollection $\{g_k:k=1,\ldots,n\}$ such that $\bigvee_{k=1}^{n}(\bar{g}_k)^o=1_X$. Now for each $k=1,2,\ldots,n$ there exists a $f_k\varepsilon$ A such that $g_k\leq f_k$. Therefore we have $(\bar{g}_k)^o\leq(\bar{f}_k)^o=f_k$ and $\bigvee_{k=1}^{n} f_k=1_X$.

(iii $\Rightarrow$iv). Let $\{g_i\}_{i\varepsilon I}$ be a collection of fuzzy regularly closed sets with the finite intersection property and suppose that $\bigwedge_{i\varepsilon I} g_i=o_X$. Then $\{1-g_i\}_{i\varepsilon I}$ is a collection of fuzzy regularly open sets with the finite intersection property and by assumption there exists a finite subset $F\subseteq I$ such that $\bigvee_{i\varepsilon F}(1-g_i)=1_X$. This implies $\bigwedge_{i\varepsilon F}g_i=o_X$, which is a contradiction. Hence $\bigwedge_{i\varepsilon I} g_i\neq o_X$.

(iv $\Rightarrow$v) . Let $\{g_i\}_{i\varepsilon I}$ be a collection of fuzzy closed sets having the given property. Then $\{(\overset{o}{g_i})^{-}\}_{i\varepsilon I}$ is a family of fuzzy regularly closed fuzzy sets having the finite intersection property. By (iv), $\bigwedge_{i\varepsilon I}(\overset{o}{g_i})^{-}\neq 0_X$. But, from $(\overset{o}{g_i})^{-}\leq g_i$ we have $\bigwedge_{i\varepsilon I} g_i\neq 0_X$.

(v $\Longrightarrow$ i). Let $\{f_i\}_{i\in I}$ be a fuzzy open cover of X. If $\bigvee_{i\in F} (\bar{f}_i)^o$ does not cover X for every finite subcollection $\{f_i\}_{i\in F}$, then $\bigwedge_{i\in F} ((1-f_i)^o)^- \neq o_X$. By (v), $\bigwedge_{i\in I} (1-f_i) \neq o_X$, i.e. $\bigvee_{i\in I} (1-(1-f_i)) \neq 1_X$ and hence the contradiction $\bigvee_{i\in I} f_i \neq 1_X$.

Obviously every nearly compact fuzzy topology is almost compact. The reverse implication does not hold in general:

2.2. EXAMPLE. Now let $X=\{a,b,c,d\}$ and τ_X be the fuzzy topology with subbase

$$\{ f_n, g_n, h_n, k_n : n=1,2,3,\ldots \}$$

where

$$f_n(a)=1-\frac{1}{n}, \; f_n(b)=1-\frac{1}{n}, \; f_n(c)=\frac{1}{2}, \; f_n(d)=1-\frac{1}{n},$$

$$k_n(a)=\frac{1}{n}, \; k_n(b)=\frac{1}{2}, \; k_n(c)=0, \; k_n(d)=0,$$

$$h_n(a)=0, \; h_n(b)=0, \; h_n(c)=\frac{1}{2}, \; h_n(d)=\frac{1}{n},$$

$$g_n(a)=1-\frac{1}{n}, \; g_n(b)=\frac{1}{2}, \; g_n(c)=1-\frac{1}{n}, \; g_n(d)=1-\frac{1}{n}.$$

Then (X,τ_X) is almost compact but not nearly compact.

2.3. REMARK. This example may also be used to show that one cannot replace "base" by "subbase" in Theorem 2.1. (ii). For consider the subbase $\{p_n, q_n, r_n, h_n, k_n : n=1,2,\ldots\}$ where

$$p_n = (\bigvee_n (f_n \wedge g_n)) \vee f_n,$$

$$q_n = (\bigvee_n (f_n \wedge g_n)) \vee g_n,$$

$$r_n = f_n \vee g_n .$$

For this family B the condition (ii) of Theorem 2.1 is satisfied since $(\bar{p}_n)^o = (\bar{q}_n)^o = (\bar{r}_n)^o = 1_X$. However as we have noted this fuzzy topological space is not nearly fuzzy compact.

Recall that a fuzzy space (X,τ_X) is called a fuzzy semiregular space iff the collection of all fuzzy regularly open sets of X forms a base for the fuzzy topology τ_X [1] .

2.4. THEOREM. A nearly compact semiregular fuzzy topological space X is compact.

Proof. Let $\{f_i\}_{i\epsilon I}$ be an open cover of X, that is $\bigvee_{i\epsilon I} f_i = 1_X$. Since X is semiregular, $f_i = \bigvee_{j\epsilon I_i} g_j^i$, where g_j^i is a fuzzy regularly open set. But, from $\bigvee_{i\epsilon I} \bigvee_{j\epsilon I_i} g_j^i = 1_X$, there exists a finite subcollection $\{g_k : k=1,\ldots,n\}$ such that $\bigvee_{k=1}^{n} g_k = 1_X$. Now for each k=1,. ...,n there exists a f_k in $\{f_i\}_{i\epsilon I}$ such that $g_k \leq f_k$. Hence we have $\bigvee_{k=1}^{n} f_k = 1_X$ as required.

2.5. COROLLARY. A fuzzy semiregular space is nearly compact iff it is compact.

Proof. This is immediate from Theorem 2.4.

In Azad [1] some weaker forms of continuity, fuzzy semicontinuity fuzzy almost continuity and fuzzy weak continuity, are considered for the first time. For a fuzzy almost continuous function we have: A fuzzy almost continuous image of an almost compact fuzzy topological space is almost compact [3] and a fuzzy strongly continuous image of an almost compact space is compact [4] . The same holds for nearly compact spaces:

2.6. COROLLARY. An almost continuous image of a fuzzy nearly compact space is fuzzy almost compact.

2.7. COROLLARY. The image of a fuzzy nearly compact space under a strongly continuous mapping is compact.

The proofs are similar to the almost compact case, and are omitted.

A. Di Concilio and G. Gerla [3] studied products of fuzzy almost compact topological spaces and proved that in general almost compactness for fuzzy topological space is not product invariant, although if X and Y are almost compact fuzzy topological spaces and if X is product-related to Y, then their fuzzy topological product is almost compact. Recall that a fuzzy space X is product related to another fuzzy space Y, if for any fuzzy set f of X and g of Y whenever $h' \not\geq f$ and $k' \not\geq g$ implies $(h' \times 1) \vee (1 \times k') \geq f \times g$, where $h \in \tau_X$ and $k \in \tau_Y$, there exists $h_1 \in \tau_X$ and $k_1 \in \tau_Y$ such that $h_1' \geq f$ or $k_1' \geq g$ and $(h_1^1 \times 1) \vee (1 \times k_1^1) = (h^1 \times 1) \vee (1 \times k^1)$ [1] where h' and k' are the complements of h and k, respectively. If f is a fuzzy set of X, g a fuzzy set of Y and X is product-related to Y, then $\bar{f} \times \bar{g} = \overline{f \times g}$ holds[1]. We omit the proof which is similar to the almost compact case.

2.8. COROLLARY. If $(X, \tau_X), (Y, \tau_y)$ are nearly compact fuzzy topological spaces and X is product-related to Y, then their fuzzy topological product is nearly compact.

A product of two nearly compact fuzzy topological spaces need not be nearly compact.

2.9. EXAMPLE. Let X be a nonempty set,

$$\tau = \{ f_\alpha \in F(X) : \alpha > \frac{1}{2} \} \cup \{ 0_X \}$$

and

$$\tau^* = \{ f_\beta \in F(X) : \beta < \frac{1}{2} \} \cup \{ 1_X \}$$

where we denote by f_α the function on X identically equal to α[3]. We have that $(\bar{f}_\alpha)^{0} = 1_X$, $\forall f_\alpha \in \tau$, $\alpha \neq 0$. Then τ is nearly compact since every open cover of τ^* must contain 1_X and then also τ^* is nearly compact. Their standard product is $\{ f_\lambda \in F(X \times X) : 0 \leq \lambda < 1 \}$. Then $(\bar{f}_\lambda)^{0} = f_\lambda$, $\forall \lambda$. Thus the open cover $\{f_\lambda\}$, $0 \leq \lambda < 1$, does not contain any finite proximate subcover.

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ÖZET

Bu çalışmada belirtisiz topolojik uzaylarda yakın tıkızlık incelendi. Belirtisiz yakın tıkızlığın belirtisiz regüler açık ve regüler kapalı kümelerle karakterizasyonu verildi.

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MATRIX BAER* - RINGS

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Let R be a ring with involution * and R_n denote the ring of all nxn matrices over R. We assume every prime homomorphic image of R has proper induced involution. Under this assumption it is proved that R_n is Baer*-ring for all n with transpose involution if and only if $P=P^*$ for all prime ideals P, the induced involution on R/P is positive definite and R is a semi-hereditary ring.

Key Words: Baer*-rings, Semi-hereditary rings, Ring of quotients

1980 Subject Classification: 16A28

1. INTRODUCTION

Throughout R will denote an associative ring with identity 1. We follow the terminology of [1] in general. If R is a ring with an involution *, we may define an involution on the nxn matrix ring R_n, by applying both the transpose, and the involution to each of its entries.

1.1. DEFINITION. Let R be a ring with involution *. The involution is positive definite if, for all finite subsets $\{r_i\}$ of R, $\Sigma r_i r_i^* = 0$ implies all the r_i are zero. An element r of R is said to be bounded if there exists a positive integer k such that $r^*r < k1$. The set of all bounded elements is denoted by R_b; it is called the bounded subring of R.

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1.2. LEMMA ([3]). For a ring R with involution, for any positive integer n, $(R_n)_b=(R_b)_n$.

We record some other results which are known for the sake of completeness.

1.3. LEMMA ([3]). Let R be any ring with positive definite involution, and let Q be its maximal right quotient ring. Suppose the involution extends to Q. Then the involution in Q is positive definite, Q_n is Baer*-ring for all n, and R_n in Baer*-ring for all n if and only if Q_b is contained in R.

1.4. LEMMA ([2]). Let R be a ring with involution *. Suppose for all $x \in R$, $1+xx^*$ is invertible in R. Then for all maximal two-sided ideals M of R, $M=M^*$.

Proof. Assume $M \neq M^*$. Since M is maximal, then the canonical mapping from $R/M \cap M^*$ to $R/M \times R/M^*$ is onto. For $(-\bar{1},\bar{1})$ in $R/M \times R/M^*$ there corresponds an $x \in R$ such that $(\bar{x},\bar{x})=(-\bar{1},\bar{1})$ holds. This implies $1+x \in M$, $1-x \in M^*$. Hence $1+x$ and $1-x^*$ lie in M and so $1+xx^* = 1+x-x(1-x^*)$ is in M. This and the invertibility of $1+x x^*$ leads us to a contradiction. Thus $M=M^*$.

2. RESULTS

In [1], §55, it is noted that "We are then left with the problem of determining conditions on a finite Baer*-ring R that are sufficient to ensure that R_n is a Baer*-ring. The problem is largely open". It is proved in [1] that R_n is Baer*-ring under severe hypothesis but AW*-algebra case is covered. The question naturally arises, under what conditions does the matrix ring R_n become a Baer*-ring. This question is largely studied in ([3],[4]).

In this note among other things, we generalize Theorem IV.5 in [2] to prime ideals.

Theorem ([2], Theorem IV.5). Let R be a semi-prime PI Goldie ring with involution * . Then R_n is Baer* with respect to *-transpose for all n, if and only if

(a) $P=P^*$ for all primitive ideals of R

(b) The induced involution on R/P is positive definite for all primitive ideals P

(c) R is semihereditary.

We begin with the following

2.1. LEMMA. Let R be a ring with involution*. Suppose R has a two sided prime ideal P satisfying

$$x\,x^* \in P \text{ implies } x \in P \quad (x \in R).$$

Then $P=P^*$.

Proof. Assume we have an $x \in P^*$ and $x \notin P$. Then for all $r \in R$, $(rx)(x^*\ r^*)=(rx)(rx)^* \in P^*$ and $(rx)(rx)^* \in P$. Since P is prime we obtain $rx \in P$ from hypothesis, thus $Rx \cdot P$ and so $x \in P$. This contradiction proves that $P=P^*$.

We consider the following condition in rings with involution.

(C1).. The invertibility of $1+xx^*$ implies $P=P^*$

2.2. LEMMA. Let R be a ring with a positive definite involution. Then for every $x \in R$ $1+xx^*$ is not a zero divisor.

Proof. Assume $(1+xx^*)t=0$ for some $t \in R$. We right multiply by t^* and we obtain $t^*t+(t^*x)(t^*x)^*=0$. This and the positive definiteness of involution implies $t^*t=0$, $t^*x=0$ which implies $t=0$.

2.3. PROPOSITION. Let R be a semi-prime Goldie ring with involution and assume every prime homomorphic image of R is regular ring. If R satisfies the following conditions

1) For each prime ideal P in R, $P=P^*$,
2) The induced involution on R/P is positive definite.
3) R is semi-hereditary.

Then R_n is Baer*-ring for all positive integer n.

Proof. R is semi-prime Goldie, therefore R_n is semi-prime

Goldie. The maximal ring $Q(R_n)$ of quotients of R_n is semi-simple artinian [5,(2.3.7)] , so for any subset S of $Q(R_n)$, the right annihilator r(S) is induced from the annihilator of an element a in $Q(R_n)$. Since R is semihereditary, by [3] R_n is principally projective, therefore $r(S)=r(a)=eR_n$ for some idempotent $e \in R_n$. To prove R_n a Baer*-ring it is enough to show $1+xx^*$ is invertible [6, Theo.26]. For if K is a primitive ideal in R_n then $K=P_n$ for some primitive ideal P in R [7, page 71] : Since each primitive ideal is prime ideal, 1) and 2) imply $P=P^*$ and the induced involution on R/P is positive definite. Since $(R/P)_n=R_n/P_n$ and all hypothesis of R are satisfied by R_n, then R_n/P_n has positive definite involution. Lemma 2.2 implies, in $(R/P)_n=R_n/P_n$, $1+xx^*$ is not zero divisor. Since R/P is regular, R_n/P_n is regular and therefore $1+xx^*$ is invertible in R_n/P_n. Then for each $x \in R_n$, and for every primitive ideal P_n, $1+xx^*$ is invertible in R_n/P_n. Thus $1+xx^*$ is invertible in R_n which proves the proposition.

2.4. PROPOSITION. Let R be a semi prime Goldie ring with involution *and assume every prime homomorphic image of R is von Neuman regular and R_n is Baer*-ring with respect to transpose involution for all n. If (Cl) holds for prime ideals P in R then

1) $P=P^*$ for all prime ideals P,

2) The induced involution on R/P is positive definite, for all prime ideals P in R,

3) R is semi-hereditary.

Proof. Since R is semi-prime Goldie, it has a semi-simple artinian maximal ring Q(R)=Q of quotients. Then * extends to Q and Q_n is Baer*-ring and Q_b the ring of bounded elements of Q is contained in R[3]. As in [2] we give a detailed proof that $1+xx^*$ is invertible in R. Since R_2 is Baer*-ring, we take $x \in R$ and we put $X=\begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}$, and $r(X)=pR_2$ for some projection p in R_2. Denote $Y=\begin{pmatrix} -x & 0 \\ 1 & 0 \end{pmatrix} \in R_2$ and then XY=0, so $Y \in r(X)$ and pY=Y and Xp=0. Since R_2 has transpose involution then p must have the from $\begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ where $a=a^*$. We use

$Xp=0$ and $pY=Y$ and we obtain $a+xb^*=0$, $-ax+b=-x$. We apply the involution to the second and use the first equality to obtain $a=xx^*(1-a)$. We add $1-a$ to both sides this equality and we conclude that $1+xx^*$ is invertible. We here invoke (CI) to get $P=P^*$ for every prime ideal P of R, and so 1) follows. Let P be a prime ideal of R, then R/P has an induced involution, and since it is regular, then R/P is a $*$-regular ring [6,Theo.26] and so $xx^* \in P$ implies $x \in P$. Since R_n/P_n satisfies all the hypotheses of R then $xx^* \in P_n$ implies $x \in P_n$ for all prime ideals P_n in R_n. Now we prove the induced involution of R/P is positive definite. For this it suffices to show $\Sigma x_i x_i \in P$ implies $x_i \in P$ where $\{x_i\} \subseteq R, (i=1,2,\dots,t)$.
Set

$$X=\begin{bmatrix} x_1 & x_2 & \cdots & x_t \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \text{then } X^*=\begin{bmatrix} x_1^* & 0 & \cdots & 0 \\ x_2^* & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ x_t^* & 0 & \cdots & 0 \end{bmatrix} \text{ and}$$

$$XX^*=\begin{bmatrix} \Sigma x_i x_i^* & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in P_n \quad \text{since } \Sigma x_i x_i^* \in P.$$

Hence $X \in P_n$ which implies $x_i \in P$ $(i=1,i,\dots,t)$. It completes the proof of 2). As for 3), R_n is Baer*-ring hence the right annihilator of any subset of R_n is generated by a projection (idempotent), and therefore R_n is principally projective, thus [3] implies that R is semihereditary.

2.5. THEOREM. Let R be a semi-prime PI Goldie ring satisfying (CI) for all prime ideals with involution. Then R_n is Baer*-ring with respect to the transpose involution for all n, if and only if

1) For all prime ideals P, we have $P=P^*$,
2) The induced involution on R/P is positive definite for all

prime ideals P

3) R is a semi-hereditary ring.

Proof. Assume R_n is a Baer*-ring for all n. As in the proof of Proposition 2.4. $1+xx^*$ is invertible and so (Cl) implies $P=P^*$ for all prime ideals. Then R/P has an induced involution and satisfies a PI. Therefore R/P has a maximal ring of quotients Q (R/P) which is semi-simple artin and von Neuman regular. Since the involution of R/P is extended to Q(R/P), Q(R/P) is a $*$-regular ring which implies Q(R/P) is regular ring with proper involution. Assume $x\, x^* \varepsilon P$ then $\overline{x\, x^*}=\bar{x}\, \bar{x}^* =\bar{0}$ in $R/P \subset Q(R/P)$ so we have $\bar{x}=\bar{0}$ which implies $x \varepsilon P$. We proceed as in the proof of Proposition 2.4. to complete the proof of 2) and 3).

Assume 1), 2) and 3) hold. In the light of Proposition 2.3, it is enough to prove $1+xx^*$ is invertible for each x in R. For if P is a primitive ideal, P is a prime ideal hence $P=P^*$ and R/P has induced involution and satisfies a PI. Therefore it is simple ring by Kaplanskky's Teorem, and it is artinian, hence R/P is regular. Thus Proposition 2.3. completes the proof of the Theorem.

2.6. THEOREM. Let R be a ring with involution* . Suppose there exists r in R and a polynomial $g(\lambda)\varepsilon Z[\lambda]$, where $Z=Z(R)$ the center of R, such that $g(r)=0$, $g'(r)$ is right invertible (where $g'(\lambda)$ is the formal derivative of $g(\lambda)$), and the centralizer of r in R is semi simple artinian. If R satisfies one of the following conditions:

i) R is semi-prime Goldie ring,

ii) R is semi-prime Goldie and an algebra over a field,

iii) R is a PI C^*-algebra,

Then R_n is Baer*-ring for all n if and only if

1) For all prime ideals P, $P=P^*$

2) The induced involution on R/P is positive definite, for all prime ideals P,

3) R is a semi-hereditary ring

Proof. Each prime homomorphic image of R is simple artinian in the case ii). In the case i) each prime image is simple artinian. In these cases the Theorem is clear. Case iii). R is semi-prime algebra and the hypothesis implies that each prime homomorphic image of R is simple artinian [8] . Hence each prime ideal P and primitive ideal P is maximal, therefore R/P is a regular ring. Hence the proofs of previous Propositions 2.3 and 2.4 carry over verbatim and we complete the proof.

ÖZET

Bu çalışmada involüsyonlu halkaların matris halkalarının Baer* halka olması için gerek ve yeter koşullar araştırıldı. R halkasının matris halkalarının Baer*-halka olması için R semi-prime Goldie, her asal ideali P için P=P*, R/P pozitif definite involüsyonlu ve R nin semi-hereditary şartlarını sağlamasının gerek ve yeter olacağı, R nin PI halka; R nin C*cebir olması durumlarında ispatlandı.

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SOME REMARKS ON THE COMMUTATIVITY OF RINGS

A. Harmancı[1]

Let R be a ring with identity. For any elements x,y in R we consider the following relations:

i) $[x^n,y]-[x,y^n]\varepsilon Z(R)$

ii) $[x^m,y]-[x,y^m]\varepsilon Z(R)$

where m,n are relatively prime integers

iii) $(xy)^2 - x^2y^2\varepsilon Z(R)$

iv) $(xy)^3 - x^3y^3\varepsilon Z(R)$

v) $[x^3,y] - [x,y^3]\varepsilon Z(R)$

In this article we prove the commutativity of semi-prime rings satisfying i) and ii) or iii) or iv). v) gives the commutativity of the ring provided that R^+ is 6- torsion free.

Key words: Commutativity, Commutator, Engel condition

1980 Subject Classification: 16A70

1. INTRODUCTION

Let R be an associative ring with identity. It is proved in[3] that if R satisfies the identities

$$[x^n,y] = [x,y^n]\ ,\ [x^{n+1},y] = [x,y^{n+1}]$$

for all x,yεR and a fixed integer n>1, then R is commutative. Recently Gupta [2] generalized this result and proved the commutativity of semiprime rings with identity satisfying

$$[x^n,y]-[x,y^n]\ \varepsilon\ Z(R),\quad [x^{n+1},y]-[x,y^{n+1}]\ \varepsilon\ Z(R)$$

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for all $x,y \in R$ and a fixed integer $n>1$, where $Z(R)$ denotes the center of R.

2. RESULTS

Throughout this paper $[x,y]$ shall denote the commutator $xy-yx$ and $C(R)$ the commutator ideal of R.

We begin with the following theorem which is a generalisation of Theorem 3 *in [2]* and Theorem B in [3] .

2.1. THEOREM. Let R be a semi-prime ring with identity satisfying

$$\text{(i)} \quad [x^n,y] - [x,y^n] \in Z(R)$$
$$\text{(ii)} \quad [x^m,y] - [x,y^m] \in Z(R) \text{ for all } x,y \text{ in } R$$

where m and n are relatively prime integers. Then R is commutative.

Proof. We first assume that R is a prime ring. We remark that in a prime ring R, $y \in Z(R)$, $x \neq 0$ and $xy \in Z(R)$ implies $x \in Z(R)$. We replace x by $1+x$ in the condition (i) and substract it from (i).We obtain by using the previous remark

$$(1) \quad n[x,y] + \sum_{k=2}^{n-1} \binom{n}{k} [x^k,y] \in Z(R).$$

Similarly, we replace x by $1+x$ in the condition ii) and we obtain

$$(2) \quad m[x,y] + \sum_{k=2}^{m-1} \binom{n}{k} [x^k,y] \in Z(R).$$

Since m and n are relatively prime, we can find integers p and q satisfying

$$(3) \quad pn+qm=1.$$

We use p and q in (1) and (2) respectively and we obtain

$$(4) \quad pn\,[x,y] + p \sum_{k=2}^{m-1} \binom{m}{k} [x^k,y] \in Z(R).$$

(5) $qm[x,y] + q\sum_{k=2}^{m-1} \binom{m}{k} [x^k,y] \in Z(R)$.

We add (4) to (5) and we invoke (3) and then we obtain

(6) $[x^2f(x)-x,y] \in Z(R)$.

As in [2] we replace y by yx in (6) to get

(7) $[x^2f(x)-x,y]\, x \in Z(R)$.

If for all x and y in R $[x^2f(x)-x,y]=0$ then it is well known that R is commutative [6]. Assume $[x^2f(x)-x,y] \neq 0$ for some x and y in R. Then since R is prime ring, the remark in the first paragraph of the proof, together with (7), implies $x \in Z(R)$. This leads us to a contradiction if $[x^2f(x)-x,y] \neq 0$. Hence for all x,y in R.

$$[x^2f(x)-x,y]=0$$

proving R to be commutative.

Assume now R is semi-prime ring. Then $P(R)=0$, the prime radical of R, and if P is any prime ideal in R, then $C(R) \subset P$ since R/P is commutative as the prime homomorphic image of R. Hence $C(R) \subset P(R)=(0)$ proving R to be commutative.

2.2. PROPOSITION. Let R be a ring with identity 1.

Assume R is a prime ring satisying the condition

(8) $(xy)^n - x^n y^n \in Z(R)$ for all $x,y \in R$, $n>1$ fixed integer. Then R has no zero divisors.

Proof. We claim first that R does not contain nilpotent elements. For if $x^k=0$ and $x^{k-1} \neq 0$, then

$$(x^{k-1}y)^n \in Z(R), \text{ So } (x^{k-1}y)^n x=0 \text{ and } (x^{k-1}y)^{n+1}=0.$$

Hence the right ideal $x^{k-1}R$ is nil of bounded nilpotency index. Lemma 1.1 [5] implies $x^{k-1}=0$ since R is a prime ring. This contradiction proves the claim.

Assume xy=0. Then yx=0. Hence xry=0 for all $r\in R$. Since R is prime x=0 or y=0. Thus R has no zero divisors.

Let R be a prime ring with identity satisfying (8) in the Proposition. For any $x\neq 0$, and y in R, commute (8) by x and left canceal x to obtain

(9) $(yx)^n - x^{n-1}y^n x \in Z$.

We interchange x and y in (8) to give

(10) $(yx)^n - y^n x^n \in Z$.

We use (9) and (10) and we get

(11) $y^n x^n - x^{n-1}y^n x = [y^n, x^{n-1}]\, x \in Z$.

We apply the condition (11) to prime rings for n=2 and n=3 and we prove the corollaries.

2.3. COROLLARY (GUPTA [2]) Let R be a prime ring with identity 1 satisfying

(*) $(xy)^2 - x^2y^2 \in Z(R)$ for all $x, y \in R$.

Then R is commutative.

Proof. For n=2, the condition (8) in the proposition takes the form (*) hence (11) gives rise to

(12) $[y^2, x]\, x \in Z(R)$.

Since R has no zero divisiors

$$[y^2, x]\, x \in Z(R) \text{ implies } [y^2, x]\, x = x[y^2, x].$$

Assume Ch R=2. Replace y by x+y in (12) to obtain

$[[x,y],x]\, x \in Z(R)$. Hence R satisfies the 3-rd Engel condition

$[[[x,y],x],x] = 0$.

If $ChR\neq 2$, Replace y by 1+y in (12) to get $[y,x]\, x \in Z(R)$. Thus

$[[x,y],x]=0$ which is the 2-nd Engel condition. In both cases R satisfies the finite Engel condition. Hence R is commutative [4].

2.4. COROLLARY . Let R be a prime ring with identity satisfying

(**) ... $(xy)^3-x^3y^3 \in Z(R)$ for all $x,y \in R$. Then R is commutative.

Proof. For n=3,(4) takes the form

(13)... $[y^3,x^2]x \in Z(R)$.

Assume ChR=2. Replace y by 1+y in (13)and we obtain $[y^2+y,x^2]x \in Z(R)$. In this we replace y by x+y and we get $[[x,y],x^2]x \in Z(R)$. In this, replace x by 1+x we have $[[x,y],x^2] \in Z(R)$. This and primeness of R implies $x \in Z(R)$.

If ChR≠2, replace x by 1+x in (13)two times to get

(14) $[y^3,x] \in Z(R)$.

We replace y by yx in (14)and use primeness of R to conclude that $x^3 \in Z(R)$. This implies the commutativity of R as it is well known.

2.5. THEOREM. Let R be a semi-prime ring with identity 1 and satisfiying one of the following conditions

(i) $(x\,y)^2-x^2y^2 \in Z(R)$

(ii) $(xy)^3-x^3y^3 \in Z(R)$ for all x,y in R.

Then R is commutative.

Proof. Every prime homomorphic image of R satisfies the conditions (i) and (ii), and then we combine the Corollaries 2.3 and 2.4 with (i) and (ii) to obtain the commutativity of R.

Note that in the general case n≥4 it is not known if condition (11)implies the commutatity of R. Some restrictions on the rings are needed since nilpotent non-commutative rings of bounded nilpotent index t>4 satisfy (11). The ring remarked on by Bell in [1] shows that some restrictions are needed to obtain commutativity for rings satisfying the condition,

$[x^n,y] - [x,y^n] \in Z(R)$ for $n \geq 2$. In this direction Bell proved the commutativity of rings R satisfying $[x^n,y]=[y,x^n]$ if R^+ is n-torsion free. For n=2 Gupta generalized this to semi-prime rings satisfying $[x^2,y]-[y,x^2] \in Z(R)$.

As in the previous proofs, by using the Engel condition for prime rings, we may prove

2.6.THEOREM . Let R be a semi-prime ring with identity satisfying

$$[x^3,y]-[x,y^3] \in Z(R), \text{ for all } x,y \text{ in } R.$$

If R^+ is 6-torsion free then R is commutative.

ÖZET

R birimli bir halka olsun. R de aşağıdaki bağıntıları göz önüne alalım:

i) $[x^n,y] - [x,y^n] \in Z(R)$

ii) $[x^m,y] - [x,y^m] \in Z(R)$

burada m,n aralarında asal pozitif tam sayılar,

iii) $(xy)^2 - x^2y^2 \in Z(R)$

iv) $(xy)^3 - x^3y^3 \in Z(R)$

v) $[x^3,y] - [x,y^3] \in Z(R)$.

Bu çalışmada yukarıdaki i) ve ii) yada iii) yada iv) bağıntılarını sağlayan halkanın komütatifliği ispatlandı. v) bağıntı halkanın, R^+ grubunun karakteristiği altı olmaması durumunda, komütatifliğini gerektirdiği gösterildi.

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POWER COMPARISONS OF SOME OUTLIER TESTS

H.Tatlıdil (1)

In this paper test statistics introduced by Gentleman - Wilk, Cook and Andrews - Pregibon for detecting outliers in regression models are considered. The critical values for these tests are obtained and the powers of the tests are compared by performing Monte Carlo technique for various sample sizes and probability levels. The performances of the underlying test procedures are also demonstrated by using the approximated percentiles in a numerical example.

Key words : Outlier, Power of a test, Monte Carlo technique

1.INTRODUCTION

A common problem in regression analysis is the detection of outliers in the data set. It is well known that an outlier usually provides a large residual when the chosen model is fitted to the data. Therefore, most of the outlier detection procedures are based on residuals or some functions of residuals such as studentized residuals and standardized residuals [2].

In this paper some outlier detection tests which are mostly based on residuals are examined. These procedures have been introduced by Gentleman and Wilk [8], Cook [3, 4], Andrews and Pregibon [1]. The statistics of these tests are respectively called QK, C_{ij} .. and AP(AP1, AP2). At first their percentage points are obtained by Monte Carlo generations as in [6,9,11], and these values are tabulated. In the second step of the simulation, the powers of the

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tests are compared against each other when a single contaminant exist for various p, n and α values. At last, the performances of these procedures are examined applied to a numerical example.

2.SOME MATRIX NOTATIONS

Let Y be an nx1 vector of response, X is an nxp matrix of explanatory variables with rank p, β is a px1 vector of unknown parameters and ε is an nx1 vector of residuals. Let also $Y = X\beta + \varepsilon$ be a multiple regression model. In usual notation, the basic model for the case when some outliers exist can be expressed as ;

$$E(Y) = E\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \beta \qquad ..(2.1)$$

Here the observations are divided into two groups, Y_2 consists of k observations which are being considered as possible outliers or influential observations, Y_1 consist of the remaining n-k observations [7, 10]. The least squares residuals of this model is given by,

$$\varepsilon = Y - Xb = (I-R)Y = \begin{bmatrix} I-R_{11} & -R_{12} \\ -R_{21} & I-R_{22} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \qquad ..(2.2)$$

where $b = (X'X)^{-1} X'Y$ and $R_{ij} = X_i(X'X)^{-1} X_j'$ is a submatrix of $R = X(X'X)^{-1} X'$ [7, 10].

Deleting the suspicious Y_2 observations gives the model $E(Y_1) = X_1\beta$. On the other hand Draper and John [7] offered an alternative model to model (2.2) by using the matrix form,

$$E\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} X_1 & 0 \\ X_2 & I \end{bmatrix} \begin{bmatrix} \beta \\ \gamma \end{bmatrix} \qquad ..(2.3)$$

where γ is a kx1 vector of additional parameters. The estimators of β and γ are respectively b^* and c, defined by,

$$b^* = (X_1' X_1)^{-1} X_1' Y_1 \qquad ..(2.4)$$

$$c = (I-R_{22})^{-1} e_2 . \qquad .(2.5)$$

Then extra sum of squares or outlier sum of squares has been shown by Gentleman and Wilk [8] and also Draper and John [6,7,9] to be

$$QK^{(2)} = e_2' (I-R_{22})^{-1} e_2 . \quad ..(2.6).$$

Estimate of σ^2 is defined by,

$$\hat{\sigma}^2 = s^2 = e'e/(n-p) = RSS/(n-p) \quad ..(2.7).$$

3.SOME EFFICIENT OUTLIER TESTS

In this section we explain briefly the test statistics that are included in the comparison.

3.1.Gentleman and Wilk Test : Gentleman and Wilk [8] developed a test statistic QK for two-way tables. Then John and Draper [9] used this statistic for one, two and three outlier cases in two-way tables. Furthermore they tabulated percentage points of F, F^* and F^{**} (which are functions of QK) by using simulation techniques [6,9]. It has been shown that QK is an efficient outlier procedure and distributed as $\sigma^2 \chi_k^2$ under the null hypothesis [8]. John and Draper also used the following statistic ;

$$F = (n-p-k)QK/k(e_2' e_2 - QK) \quad ..(3.1)$$

as a test criterion. F has a central F distribution with k and n-p-k degrees of freedom [6].

3.2.Cook Test : Cook [3,4] proposed a test statistic based on confidence ellipsoids for judging the contribution of each data point to the determination of the least squares estimate of β. This statistic is defined by,

$$C_{ij} = (b-b^*)'X'X(b-b^*)/ps^2 \quad ..(3.2)$$

where b and b^* denote the estimates of β with and without the ith and jth data points. ith and jth elements of Y belong to the Y_2 response vector and X_2 submatrix. Cook and Weisberg [5] considered the performance of this test statistic. Draper and John [7]

(2) QK values given in this study are 1/100 of their actual values.

expressed $C_{ij..}$ as,

$$C_{ij..} = c'R_{22}c/ps^2 \quad ..(3.3),$$

and because of the equality $c'R_{22}c = c'c-QK$, $C_{ij..}$ is

$$C_{ij..} = \frac{QK}{ps^2}\left[\frac{c'c}{QK} - 1\right] \quad ..(3.4).$$

3.3. Andrews and Pregibon Test : The Andrews and Pregibon statistic AP is based on matrix of $X^* = (X:Y)$, the matrix of explanatory variables appended with Y vector. This matrix for model in equation (2.1) is,

$$X_1^* = (X:Y) \quad ..(3.5),$$

and for model in equation (2.3) is

$$X_2^* = (X:D:Y) \quad ..(3.6)$$

where $D = \begin{bmatrix} 0 \\ I \end{bmatrix}$. The Andrews-Pregibon statistic AP is defined as ;

$$R^k_{ij..} = |X_2^{*\prime}X_2^*| / |X_1^{*\prime}X_1^*| \quad ..(3.7).$$

ij.. denote the k subscripts selected to form Y_2 [1]. Test statistic is based on the proportion of volume in X_1^* attributable to the k observations which are possible outliers. Draper and John [7] established that,

$$R^k_{ij..} = (1-QK/RSS)\,|I-R_{22}| = AP1xAP2 = AP \quad ..(3.8).$$

where AP1= (1-QK/RSS) and AP2= $|I-R_{22}|$ also proposed as two new outlier tests. Then Tatlıdil [12] showed that AP1 has a beta distribution with (n-p-k)/2 and k/2 degrees of freedom under the assumption that AP1 and AP2 are independent. Tatlıdil also showed that AP1 and AP (which has a central beta distribution with (n-p-1)/2 and p/2 degrees of freedom) tests can be used for testing outliers in multivariate data. The critical values of AP and AP1 have been obtained by using first and second order Bonferroni inequalities and these values have been tabulated for k=1(1)5 ; p=2(1)9 ; n=5(1) 30(5)50(10)120 and α= 0.01,0.05,0.10 [12].

4. SIMULATION STUDY

In this section, the percentage points of the test statistics QK, AP1, AP2, AP and C_{ij} are obtained for each combination of p(number of parameters) and n(sample size) while k(number of outlier)=1. The percentiles are estimated from the equations (2.6), (3.4) and (3.8) by using Monte Carlo techniques. 500 samples are generated for each case. Then the powers of the precedures are obtained for the same values of the parameters.

In the simulation study, a multiple regression model as

$$Y_i = X_{i1}\beta_1 + \cdots + X_{ip}\beta_p + \varepsilon_i \qquad i=1,2,\ldots,n \qquad ..(4.1)$$

is considered and it has two main stages.

In the first stage of the simulation the following steps are employed.

i) n values of first column of X matrix are filled with 1 while other nx(p-1) values of the p-1 columns with the values 0 and 1.

ii) n values of the residual vector ε are generated from N(0,1) population.

iii) n values of the response vector Y are obtained from the equation in (4.1)

iv) Test statistics QK, C_{ij} , AP1, AP2 and AP are calculated from equations (2.6), (3.4) and (3.8).

After repeating the process 500 times, the values of each test statistic are sorted in ascending order. Then the upper percentage points of QK and C_{ij} and lower percentage points of AP1, AP2 and AP are recorded as critical values of these tests statistics. They are tabulated and given in Appendix A, Table A.1 for p=2(1)5; n=10(5)40 and α=0.01,0.05 and 0.10.

The second stage of the simulation consist of two substages. In the first substage all the above steps are employed except a difference in the second step. That is a perturbation in the nth value of the residual vector by a constant $A = \mp 1$ (if generated value e_n negative - A is added, otherwise A is added). On the other hand there is one more step, this is:

v) Calculated values of the test statistics are compared with the corresponding critical values given in Table A.1.

This process is also repeated 500 times and the number of the cases where calculated values of QK and C_{ij} are greaterthan their critical values and calculated values of AP1,AP2 and AP are less than their critical values are denoted as their powers. These values are tabulated in Appendix B, Table B.1.

In the second substage the value of the constant A is changed. In this case $A = \mp 2$. The values obtained in the second substage are given in the Table B.2, for the same parameters.

5.A NUMERICAL EXAMPLE AND DISCUSSION

We demonstrate the test procedures considered in this study by using a data set given by Mickey, Dunn and Clark and examined by Draper and John [7] and Little [10]. The observations and their corresponding values of the test statistics are given in the Table 1.

It is seen from the table that observation 19 is identified as an outlier if QK and AP1 tests are used, whereas observation 18 is identified outlier if AP2, AP and C_{ij} tests are used. Furthermore the correlations between the values of the test statistics(columns of Table 1) are :

COR.MATRIX =

Variable	AP1	AP2	AP	C_{ij}
QK	-0.999	0.053	-0.539	0.295
AP1		-0.053	0.539	-0.294
AP2			0.812	-0.934
AP				-0.960

The results of the outlier analysis and also the correlation

TABLE 1.Age at First Word(X), Gessel Adaptive Score(Y) and their corresponding Test Statistics Values.

Case	X	Y	QK	AP1	AP2	AP	C_{ij}
1	15	95	0.043	0.998	0.952	0.950	0.009
2	26	71	1.084	0.953	0.845	0.806	0.081
3	10	83	2.598	0.887	0.937	0.832	0.072
4	9	91	0.820	0.964	0.929	0.896	0.026
5	15	102	0.857	0.963	0.952	0.917	0.017
6	20	87	0.001	1.000	0.927	0.927	0.000
7	18	93	0.124	0.995	0.942	0.937	0.003
8	11	100	0.067	0.997	0.943	0.941	0.002
9	8	104	0.107	0.995	0.920	0.916	0.004
10	20	94	0.479	0.979	0.927	0.908	0.015
11	7	113	1.334	0.942	0.909	0.857	0.055
12	9	96	0.150	0.994	0.929	0.923	0.005
13	10	83	2.598	0.887	0.937	0.832	0.072
14	11	84	1.925	0.917	0.943	0.865	0.048
15	11	102	0.217	0.991	0.943	0.934	0.005
16	10	100	0.021	0.999	0.937	0.936	0.000
17	12	105	0.789	0.966	0.948	0.915	0.018
18	42	57	0.881	0.962	0.348*	0.335*	0.678*
19	17	121	9.686*	0.580*	0.947	0.550	0.223
20	11	86	1.396	0.939	0.943	0.886	0.035
21	10	100	0.021	0.999	0.937	0.936	0.000
Unusual values are			High	Low	Low	Low	High

matrix showed that QK and AP1 statistics tends to provide similar results for outliers. In the discussions of the various authors such as Little [10], observation 19 was also identified as an outlier. AP2, AP and C_{ij} statistics approximately provide the similar results which was also mentioned in the previous works. These statistics are sensitive to influential observations which have much affect on the fitted equation. As a conclusion QK is the most powerful test among them.

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APPENDIX A : CRITICAL VALUES OF TEST STATISTICS

TABLE A.1. Critical Values of QK, AP1, AP2, AP and C_{ij} Statistics Obtained from 500 Generations for Various Values of n and p.

	P:	2			3			4			5		
Test	n α:	0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10
	10	3.424	1.981	1.608	3.824	2.172	1.678	4.669	2.700	2.258	3.024	2.527	1.902
	15	3.783	1.986	1.542	3.217	2.961	1.542	3.616	2.685	1.845	3.392	2.730	2.027
	20	3.881	1.895	1.365	2.447	2.098	1.456	3.892	2.078	1.701	4.091	2.669	2.103
QK	25	3.588	2.678	1.648	4.599	2.486	1.984	4.165	3.208	2.131	3.963	2.466	1.772
	30	2.440	2.148	1.803	3.851	1.874	1.600	4.340	2.732	1.524	2.987	1.868	1.436
	35	3.617	2.328	1.840	3.345	2.553	2.052	3.096	2.562	1.649	3.533	2.039	1.597
	40	3.504	1.872	1.648	2.979	1.678	1.407	3.338	2.855	1.851	4.476	2.503	1.754
	10	0.389	0.589	0.649	0.437	0.563	0.660	0.197	0.412	0.475	0.301	0.410	0.466
	15	0.604	0.750	0.837	0.508	0.729	0.770	0.644	0.701	0.774	0.434	0.564	0.687
	20	0.692	0.847	0.896	0.724	0.801	0.852	0.662	0.785	0.829	0.669	0.725	0.825
AP1	25	0.693	0.840	0.894	0.757	0.828	0.860	0.677	0.769	0.791	0.657	0.824	0.876
	30	0.839	0.868	0.889	0.770	0.837	0.836	0.727	0.826	0.892	0.780	0.895	0.908
	35	0.843	0.883	0.923	0.867	0.909	0.932	0.850	0.873	0.915	0.779	0.837	0.904
	40	0.900	0.917	0.938	0.859	0.905	0.919	0.846	0.884	0.912	0.813	0.891	0.908
	10	0.423	0.543	0.579	0.387	0.428	0.513	0.282	0.343	0.392	0.137	0.187	0.277
	15	0.545	0.612	0.718	0.445	0.469	0.571	0.352	0.450	0.498	0.379	0.494	0.523
	20	0.685	0.741	0.768	0.565	0.640	0.682	0.501	0.593	0.625	0.528	0.581	0.614
AP2	25	0.702	0.817	0.839	0.623	0.715	0.777	0.571	0.657	0.692	0.623	0.680	0.732
	30	0.773	0.829	0.881	0.620	0.711	0.769	0.644	0.725	0.769	0.679	0.725	0.753
	35	0.770	0.829	0.862	0.681	0.764	0.830	0.769	0.801	0.839	0.704	0.756	0.778
	40	0.839	0.875	0.889	0.677	0.743	0.796	0.805	0.818	0.831	0.776	0.800	0.809

TABLE A.1. (Continued)

Test	n	P:	2			3			4			5		
		α:	0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10
AP	10		0.247	0.416	0.456	0.264	0.370	0.413	0.108	0.174	0.226	0.074	0.146	0.169
	15		0.513	0.561	0.631	0.369	0.448	0.492	0.326	0.370	0.440	0.264	0.352	0.381
	20		0.618	0.671	0.728	0.449	0.565	0.638	0.434	0.523	0.552	0.459	0.500	0.541
	25		0.665	0.780	0.809	0.511	0.623	0.718	0.478	0.548	0.646	0.452	0.605	0.663
	30		0.746	0.808	0.819	0.619	0.647	0.731	0.606	0.665	0.705	0.636	0.669	0.706
	35		0.743	0.791	0.813	0.625	0.727	0.819	0.711	0.764	0.776	0.637	0.685	0.728
	40		0.803	0.852	0.861	0.629	0.743	0.796	0.756	0.779	0.790	0.658	0.744	0.767
C_{ij}	10		1.413	0.688	0.487	1.154	0.473	0.308	1.463	1.037	0.790	2.906	0.900	0.666
	15		0.501	0.258	0.148	0.916	0.374	0.298	0.747	0.471	0.349	0.870	0.518	0.383
	20		0.404	0.226	0.163	0.884	0.288	0.136	0.551	0.361	0.205	0.407	0.362	0.216
	25		0.165	0.104	0.076	0.430	0.282	0.135	0.553	0.408	0.237	0.426	0.206	0.121
	30		0.160	0.128	0.089	0.274	0.147	0.092	0.300	0.164	0.118	0.298	0.150	0.105
	35		0.288	0.170	0.110	0.155	0.118	0.096	0.170	0.125	0.081	0.296	0.183	0.096
	40		0.160	0.093	0.080	0.136	0.064	0.048	0.157	0.129	0.091	0.309	0.143	0.103

APPENDIX B : POWERS OF TESTS OBTAINED FROM MONTE-CARLO SIMULATIONS

TABLE B.1. Powers of the Tests QK, AP1, AP2, AP and C_{ij} for Various Values of n, p and α while A= 1.

	α:	0.01					0.05					0.10				
P	n Test:	QK	AP1	AP2	AP	C_{ij}	QK	AP1	AP2	AP	C_{ij}	QK	AP1	AP2	AP	C_{ij}
	10	0.20	0.10	0.02	0.05	0.04	0.46	0.35	0.06	0.26	0.15	0.59	0.47	0.11	0.32	0.20
	15	0.23	0.18	0.00	0.18	0.16	0.55	0.43	0.02	0.44	0.42	0.65	0.68	0.13	0.49	0.55
	20	0.10	0.05	0.02	0.10	0.07	0.56	0.54	0.08	0.16	0.21	0.69	0.71	0.11	0.39	0.29
2	25	0.24	0.06	0.01	0.13	0.30	0.45	0.44	0.10	0.45	0.42	0.68	0.68	0.15	0.58	0.52
	30	0.51	0.29	0.02	0.21	0.23	0.58	0.39	0.10	0.39	0.29	0.67	0.51	0.17	0.42	0.37
	35	0.24	0.23	0.03	0.13	0.09	0.44	0.40	0.04	0.23	0.22	0.61	0.62	0.08	0.37	0.38
	40	0.18	0.34	0.03	0.16	0.15	0.55	0.43	0.07	0.37	0.35	0.65	0.59	0.09	0.46	0.40
	10	0.22	0.15	0.04	0.12	0.03	0.42	0.32	0.05	0.28	0.20	0.55	0.45	0.16	0.35	0.35
	15	0.20	0.07	0.02	0.03	0.02	0.24	0.37	0.03	0.19	0.19	0.65	0.47	0.08	0.26	0.34
	20	0.30	0.19	0.01	0.07	0.02	0.51	0.36	0.03	0.19	0.31	0.71	0.49	0.09	0.36	0.48
3	25	0.19	0.27	0.02	0.02	0.05	0.51	0.45	0.06	0.15	0.18	0.65	0.56	0.09	0.45	0.47
	30	0.27	0.07	0.01	0.05	0.19	0.62	0.35	0.01	0.11	0.42	0.65	0.49	0.12	0.33	0.57
	35	0.37	0.43	0.00	0.01	0.24	0.54	0.56	0.02	0.21	0.29	0.62	0.76	0.09	0.54	0.37
	40	0.30	0.20	0.00	0.02	0.21	0.63	0.43	0.00	0.09	0.51	0.84	0.60	0.02	0.23	0.59

TABLE B.1. (Continued)

p	n	α:	0.01					0.05					0.10				
		Test:	QK	AP1	AP2	AP	C_{ij}	QK	AP1	AP2	AP	C_{ij}	QK	AP1	AP2	AP	C_{ij}
	10		0.11	0.05	0.01	0.05	0.06	0.36	0.25	0.06	0.19	0.14	0.45	0.33	0.10	0.26	0.18
	15		0.23	0.30	0.00	0.09	0.06	0.40	0.40	0.02	0.15	0.16	0.58	0.59	0.05	0.26	0.23
	20		0.20	0.09	0.01	0.08	0.10	0.53	0.44	0.05	0.22	0.17	0.61	0.62	0.11	0.24	0.44
4	25		0.20	0.11	0.00	0.06	0.07	0.25	0.22	0.09	0.14	0.11	0.50	0.24	0.14	0.32	0.25
	30		0.20	0.07	0.01	0.12	0.13	0.41	0.31	0.06	0.18	0.28	0.72	0.64	0.09	0.31	0.44
	35		0.30	0.24	0.03	0.13	0.22	0.35	0.33	0.11	0.34	0.26	0.64	0.54	0.22	0.37	0.43
	40		0.32	0.22	0.08	0.24	0.17	0.40	0.44	0.09	0.31	0.26	0.65	0.57	0.12	0.37	0.38
	10		0.19	0.07	0.03	0.04	0.01	0.29	0.18	0.05	0.13	0.15	0.36	0.26	0.18	0.20	0.18
	15		0.14	0.04	0.03	0.05	0.06	0.36	0.17	0.17	0.23	0.19	0.49	0.41	0.19	0.28	0.28
	20		0.16	0.17	0.07	0.17	0.12	0.35	0.28	0.13	0.26	0.18	0.48	0.57	0.20	0.32	0.29
5	25		0.12	0.05	0.06	0.11	0.10	0.41	0.37	0.17	0.22	0.20	0.57	0.58	0.26	0.36	0.41
	30		0.34	0.23	0.06	0.22	0.11	0.56	0.54	0.11	0.33	0.32	0.68	0.64	0.20	0.43	0.44
	35		0.31	0.16	0.06	0.16	0.12	0.50	0.28	0.15	0.25	0.23	0.63	0.52	0.21	0.44	0.51
	40		0.14	0.09	0.08	0.07	0.04	0.45	0.38	0.13	0.25	0.18	0.64	0.54	0.18	0.31	0.26

TABLE B.2. Powers of the Tests OK, AP1, AP2, AP and C_{ij} for Various Values of n, p and α while A= 2.

p	n	α: 0.01 OK	AP1	AP2	AP	C_{ij}	α: 0.05 OK	AP1	AP2	AP	C_{ij}	α: 0.10 OK	AP1	AP2	AP	C_{ij}
	10	0.90	0.38	0.02	0.23	0.10	0.95	0.75	0.06	0.61	0.23	0.97	0.80	0.11	0.72	0.37
	15	0.87	0.57	0.02	0.50	0.24	0.95	0.87	0.06	0.66	0.47	0.97	0.95	0.10	0.79	0.72
	20	0.81	0.54	0.00	0.52	0.27	1.00	0.99	0.00	0.90	0.69	1.00	1.00	0.01	0.93	0.85
2	25	0.96	0.45	0.00	0.54	0.49	0.99	0.95	0.09	0.92	0.64	1.00	1.00	0.12	0.97	0.70
	30	1.00	0.91	0.00	0.77	0.76	1.00	0.97	0.00	0.96	0.90	1.00	1.00	0.05	0.98	0.97
	35	0.96	0.83	0.00	0.61	0.34	1.00	1.00	0.04	0.84	0.55	1.00	1.00	0.06	0.94	0.63
	40	0.95	0.99	0.00	0.81	0.55	1.00	1.00	0.00	0.96	0.90	1.00	1.00	0.00	0.99	0.91
	10	0.63	0.46	0.04	0.38	0.06	0.85	0.67	0.04	0.62	0.44	0.89	0.80	0.16	0.72	0.57
	15	0.86	0.36	0.01	0.29	0.18	0.87	0.85	0.03	0.51	0.49	0.97	0.90	0.11	0.62	0.54
	20	0.95	0.73	0.01	0.27	0.22	0.98	0.90	0.03	0.64	0.45	1.00	0.98	0.09	0.78	0.87
3	25	0.79	0.82	0.01	0.31	0.25	0.97	0.95	0.04	0.60	0.40	1.00	0.99	0.08	0.90	0.84
	30	0.89	0.55	0.00	0.25	0.27	0.99	0.86	0.04	0.37	0.40	0.99	0.97	0.07	0.74	0.73
	35	0.90	0.92	0.02	0.25	0.61	0.99	0.98	0.04	0.62	0.71	0.99	0.99	0.12	0.94	0.77
	40	0.97	0.72	0.01	0.48	0.55	0.99	0.90	0.01	0.59	0.58	1.00	1.00	0.01	0.68	0.70

TABLE B.2. (Continued)

P	n	α: 0.01 OK	0.01 AP1	0.01 AP2	0.01 AP	0.01 C_{ij}	0.05 OK	0.05 AP1	0.05 AP2	0.05 AP	0.05 C_{ij}	0.10 OK	0.10 AP1	0.10 AP2	0.10 AP	0.10 C_{ij}
	10	0.58	0.11	0.01	0.04	0.10	0.87	0.41	0.01	0.15	0.17	0.90	0.57	0.01	0.19	0.28
	15	0.65	0.66	0.02	0.27	0.22	0.86	0.76	0.05	0.40	0.37	0.91	0.89	0.10	0.59	0.52
	20	0.75	0.61	0.02	0.32	0.36	0.95	0.83	0.05	0.55	0.48	0.98	0.93	0.11	0.67	0.65
4	25	0.74	0.45	0.00	0.22	0.23	0.91	0.81	0.09	0.45	0.27	0.97	0.86	0.14	0.73	0.54
	30	0.81	0.45	0.00	0.23	0.28	1.00	0.89	0.00	0.40	0.56	1.00	1.00	0.01	0.53	0.67
	35	0.97	0.85	0.02	0.44	0.29	1.00	0.98	0.02	0.71	0.42	1.00	1.00	0.05	0.73	0.51
	40	0.97	0.84	0.01	0.33	0.52	1.00	0.94	0.01	0.54	0.78	1.00	1.00	0.05	0.81	0.83
	10	0.53	0.30	0.03	0.09	0.02	0.64	0.54	0.05	0.33	0.24	0.73	0.61	0.18	0.40	0.32
	15	0.71	0.28	0.03	0.25	0.16	0.83	0.65	0.17	0.48	0.32	0.87	0.80	0.19	0.52	0.42
	20	0.41	0.51	0.09	0.48	0.40	0.85	0.70	0.17	0.59	0.44	0.92	0.88	0.19	0.74	0.69
5	25	0.79	0.43	0.07	0.25	0.27	0.91	0.88	0.08	0.66	0.57	0.96	0.97	0.17	0.84	0.80
	30	0.85	0.60	0.04	0.50	0.30	0.95	0.92	0.12	0.65	0.62	0.96	0.96	0.26	0.82	0.81
	35	0.87	0.56	0.08	0.45	0.25	0.97	0.81	0.12	0.59	0.46	0.99	0.99	0.15	0.73	0.74
	40	0.70	0.61	0.09	0.28	0.16	0.98	0.94	0.14	0.75	0.61	0.99	0.96	0.14	0.86	0.78

ÖZET

Bu çalışmada daha önce Gentleman - Wilk, Cook ve Andrews - Pregibon tarafından regresyon modellerindeki aykırı değerlerin (outliers) test edilmesi için önerilen yöntemler incelenmiştir. Monte-Carlo benzeşim yöntemi kullanılarak bu testlere ilişkin kritik değerler p=2(1)5; n=10(5)40 ve α=0.01, 0.05,0.10 için tablolaştırılmıştır, daha sonra ise yine aynı parametreler için yöntemlerin güç değerleri bulunarak karşılaştırılmıştır. Son olarak da bu yöntemlerin geçerlilikleri (elde edilen kritik değerler kullanılarak) sayısal bir örnek üzerinde incelenmiştir.

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PARTIAL SOLUTION FOR STACKELBERG DISEQUILIBRIUM IN DUOPOLY

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In this study, we obtained a partial solution for the Stackelberg disequilibrium situation in a duopolistic market, by assuming the sum of the profit functions of two firms as the objective function of the multicriterion decision making problem and using two-person zero-sum game approach.

In order to get better understanding of the solution an application is given in Section 4.

Key words: Stackelberg disequilibrium, Duopoly

I. INTRODUCTION

In a duopolistic industry there are two sellers. There are no generally accepted behavior assumptions for duopolists as there are for perfect competitors and monopolists. Different assumptions produce different solutions for duopolistic market. The well known solutions are Cournot, collusion, Stackelberg, market shares and kinked demand curve.

In the Stackelberg solution one duopolist is leader while the other is follower. Duopolist I is leader in the sence that he knows II's reaction function. Each duopolist determines his maximum profit levels from both leadership and followership and desires to play the role which

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yields the larger profit. Four outcomes are possible: i) I desires t be leader and II a follower, ii) II desires to be a leader and I a follower, iii) both desire to be followers or, iv) both desire to be leaders. The first two outcomes result in a determinate equilibrium which is known as Stackelberg solution. The third outcome results in Cournot's solution as shown in Henderson and Quandt [2,pp.230]. The fourth outcome is known as Stackelberg disequilibrium.

In this study a solution for the Stackelberg disequilibrium case is obtained. In the duopolistic market when each of the duopolist desires to be leader, each duopolist thinks that he knows the other' reaction pattern and determines the profit-maximizing supply based o the other's reaction function but neither of the reaction functions is observed and Stackelberg disequilibrium comes into existence. Und this condition, over-production and different prices result in the market. After they become conscious of this situation they may reach an agreement in determining the price and the profit shares to maximize profits of each other. This leads the market to a new part solution for the Stackelberg disequilibrium case. It is a partial solution and not a definite solution because it comes about only if agreement between duopolists is reached.

2. METHOD

A game with two players where a gain to one player equals a loss to the other is known as a two-person zero-sum game [3,pp.339]. Each player has a finite number of strategies. The matrix which summarize the outcomes in terms of the gain (or loss) to one player, for all possible strategies of both players is called the pay-off matrix.

The entries of the pay-off matrix π_{kt} (k= 1,...,p, t=1,...,p) represent the expected gain for player I when he uses strategy k and player II uses strategy t. Let λ_k be the probability that player I will use strategy k, and μ_t be the probability that player II will use strategy t, then

$$\sum_{k=1}^{p} \lambda_k = 1 \qquad \lambda_k \geqslant 0 \quad \text{for all } k \qquad ..(2.1)$$

$$\sum_{t=1}^{p} \mu_t = 1 \qquad \mu_t \geqslant 0 \quad \text{for all } t \qquad ..(2.2).$$

The expected pay-off for the game is given by,

$$P = \sum_{k=1}^{p} \sum_{t=1}^{p} \pi_{kt} \lambda_k \mu_t \qquad ..(2.3).$$

Let P_0 be the minimum value of P and P^0 be the maximum value of P. The solution of the game can be obtained by solving either of the following pairs of linear programming problems [1,1973]:

$$\text{Min } \left(\frac{1}{P_0}\right) = \sum_{k=1}^{p} r_k \qquad ..(2.4)$$

s.t. $$\sum_{k=1}^{p} \pi_{kt} r_k \geqslant 1, \quad r_k \geqslant 0 \qquad ..(2.5)$$

or

$$\text{Max } \left(\frac{1}{P^0}\right) = \sum_{t=1}^{p} s_t \qquad ..(2.6)$$

S.t. $$\sum_{t=1}^{p} \pi_{kt} s_t \leqslant 1 \quad , \quad s_t \geqslant 0 \qquad ..(2.7)$$

At the optimality,

$$P_0^* = P^{0^*} = P^* \qquad ..(2.8)$$

and

$$\lambda_k^* = r_k^* \; P^* \;, \qquad \mu_t^* = s_t^* \; P^* \qquad ..(2.9)$$

where P^* is optimal solution.

3. MODEL

Two firms are assumed to produce a homogenous product. The inverse demand function states the price as a function of the aggregate quantity sold:

$$p = F\,(q_1 + q_2) \qquad ..(3.1)$$

where p is price, q_1 and q_2 are quantities of the duopolists outputs. The profit of each equals his total revenue less his cost, which depends upon his output alone.

$$\pi_1 = R_1(q_1, q_2) - C_1(q_1) \qquad ..(3.2)$$

$$\pi_2 = R_2(q_1, q_2) - C_2(q_2) \qquad ..(3.3)$$

where π_1 and π_2 are profits, R_1 and R_2 revenues, C_1 and C_2 costs of the first and the second firms.

By assuming the sum of the profit functions of the firms as the objective function of the multicriterion decision making problem and using two person zero-sum game approach we may reach a partial solution for the Stackelberg disequilibrium situation.

Solving for the leadership functions of I and II individually we can determine q_i's and π_i's under Stackelberg assumptions: Forming pay-off matrix with entries of profits of the firms under leadership function of each of the firms, then using two person zero-sum game method, optimal weights λ_i's are obtained, and the compromise objective function is,

$$\pi = \sum_{i=1}^{2} \lambda_i \pi_1 (q_i) \quad ..(3.4)$$

where λ_i's are the optimal weights that firm I will use strategy i. Then

$$\lambda_1 + \lambda_2 = 1$$

$$\lambda_1, \lambda_2 \geqslant 0$$

4. APPLICATION

We use the example in Henderson and Quandt in order to get a clear explanation of the partial solution [2,pp 226-231].

$$p = 100 - 0.5\ (q_1 \quad q_2) \quad ..(4.1)$$

$$C_1 = 5\ q_1 \quad ..(4.2)$$

$$C_2 = 0.5\ q_2^2 \quad ..(4.3)$$

and the profits of the duopolists are

$$\pi_1 = 95\ q_1 - 0.5\ q_1^2 - 0.5\ q_1 q_2 \quad ..(4.4)$$

$$\pi_2 = 100\ q_2 - 0.5\ q_1 q_2 - q_2^2 \quad ..(4.5)$$

For maximizing conditions of I and II we set appropriate partial derivatives equal to zero:

$$\frac{\partial \pi_1}{\partial q_1} = 95 - q_1 - 0.5\ q_2 = 0 \quad ..(4.6)$$

$$\frac{\partial \pi_2}{\partial \pi_2} = 100 - 0.5\ q_1 - 2\ q_2 = 0 \quad ..(4.7)$$

The corresponding reaction functions are,

$$q_1 = 95 - 0.5 \ q_2 \qquad ..(4.8)$$

$$q_2 = 50 - 0.25 \ q_1 \qquad ..(4.9).$$

The maximum leadership profit of I is obtained by substituting II's reaction function (4.9) into I's profit equation(4.4),

$$\pi_1 = 70 \ q_1 - 0.375 \ q_1^2 \qquad ..(4.10)$$

and maximizing with respect to q_1 we obtain

$$q_1 = 93.33 \qquad \pi_1 = 3266.66$$

Under the leadership of I, the II's production and profit are

$$q_2 = 26.66 \quad \text{and} \quad \pi_2 = 711.11 ,$$

That is

$$q^{*1} = (93.33, 26.66) .$$

Likewise for the II's leaderahip and I's followership assumptions the profit function of II is

$$\pi_2 = 52.5 \ q_2 - 0.75 \ q_2^2 \qquad ..(4.11)$$

and the solutions for maximizing are

$$q_2 = 35 \qquad \pi_2 = 918.75$$

$$q_1 = 775 \qquad \pi_1 = 3003.125$$

that is

$$q^{*2} = (35, 77.5).$$

Each duopolist receives a greater profit from leadership, and both desire to act as leaders. Under this situation the reaction functions will never be observed and Stackelberg disequilibrium

The pay-off matrix with entries π_{ij} is formed as

	q^{*1}	q^{*2}
π_1	3266.66	3003.125
π_2	711.11	918.75

Solving for optimal profit shares as in the case of two-person zero-sum game technique, we get

$$3266.66\ \lambda_1 - 711.111\ \lambda_2 \geqslant P$$

$$3003.125\ \lambda_1 - 918.75\ \lambda_2 \geqslant P$$

where P is the value of the game, and

$$\lambda_1 + \lambda_2 = 1$$

$$\lambda_1^* = .\ 440678$$

$$\lambda_2^* = .559322$$

Maximize $\pi = \lambda_1\ \pi_1 + \lambda_2\ \pi_2$

$$= \lambda_1 (95\ q_1 - 0.5\ q_1^2) - 0.5\ q_1\ q_2$$

$$+ \lambda_2\ (100\ q_2 - q_2^2 - 0.5\ q_1\ q_2)\ .$$

For maximization

$$\frac{\partial \pi}{\partial q_1} = \lambda_1 (95 - q_1 - 0.5\ q_2) - 0.5\ \lambda_2\ q_2 = 0$$

$$\frac{\partial \pi}{\partial q_2} = -0.5\ \lambda_1\ q_1 - \lambda_2\ (100 - 2\ q_2 - 0.5\ q_1) = 0\ .$$

The solutions are

$$q_1 = 77.64705911$$

$$q_2 = 15.29411744$$

In that case we obtain the profits as

$$\pi_1 = 3768.17$$

$$\pi_2 = 701.73$$

$$\pi = 4469.2.$$

Also different solutions of the duopolistic market under different assumptions are summarized in Table 1 for comparison.

Table 1. Eguilibrium Values of the Price, Quantity Produced and Profit of the Duopolistic Market

		Collusion Solution	Cournot Solution	Stackelberg Solution[a]		New Solution
				I lead. II foll.	I foll. II lead	
Market Price		52.5	45	40	43.75	53.5[illegible]
Quantity Produced	Market	95	110	120	112.5	92.94
	Firm I	90	80	93.33	77.5	77.65
	Firm II	5	30	26.66	35	15.29
Profit	Market	4525	4100	3977.78	3921.875	4469.9
	Firm I	4275	3700	3266.66	3003.125	3768.2
	Firm II	250	900	711.11[b]	918.75	701.7

a) The entries are taken from Henderson and Quandt 2, pp 175-186.

b) This entry is different from the book.

ÖZET

Bu çalışmada, iki satıcılı piyasada, iki firmanın kār fonksiyonları toplamı, çok amaçlı karar fonksiyonu varsayılarak, iki-kişili sıfır toplamlı oyun yaklaşımı ile Stakelberg dengesizliği durumunda kısmi çözüm elde edilmiştir.

Çözümün daha iyi anlaşılabilmesi için bir uygulama da yapılmıştır.

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A DYNAMIC REGRESSION ANALYSIS OF THE ENERGY CONSUMPTION BASED ON INCOME

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Based on annual data a dynamic regression model is built for the endogenous time series Y_t = energy consumption and exogenous series x_t = per capita gross national product. Dynamic regression models of which error terms fits to AR(1), AR(2), MA(1), ARMA(1, 1) stationary stochastic processes are reviewed and maximum likelihood estimators of these models are introduced. It is concluded that the dynamic regression models are found more efficient than the classical regression model.

Key words: Dynamic Regression, ARMA(p,q) Models, ML Estimators

INTRODUCTION

One of the assumptions of the classical linear regression model is the serial independence of the disturbances, that is $E(uu') = \sigma^2 I_n$.

A general linear model can be written in matrix notation in the following manner:

$$Y = Xb + u$$

where, X is n x k matrix of the observations on explanatory variables and is assumed to be full rank, b is a k x 1 vector of parameters to be estimated, u is an n x 1 vector of stochastic disturbances. Fundamental assumptions of a normal classical linear regression model can be written out in the following

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manner: (1) Zero mean assumption: $E(u) = 0$. (2) Homoscedasticity assumption + non-autocorrelated error assumption: $E(uu') = \sigma^2 I_n$, when I_n is an n by n identy-matrix. (3) Lack of simultaneity assumption: $E(X'u) = 0$.

In practice, using ordinary least squares (QLS) estimates when disturbances of the model are autocorrelated is the possibility of non-sense relationships between time series when in reality there is no correlation between the series under investigation. This point was first emphasized by Cochrane and Orcutt [3] and was later on formalized by Champernowne [2]. In order to avoid many complications and spurious results of the QLS regression, some models are proposed; called dynamic regression models, where it is assumed that the u_t error term follows a stationary stochastic process such as autoregressive process, AR(1), first order moving average process, MA(1), second order autoregressive process, AR(2) and the first order mixed autoregressive moving average process, ARMA(1,1). A brief review of the literature on the problem of dynamic regression can be found in Paseran and Slater [7].

DYNAMIC REGRESSION MODELS

Dynamic specifications of the models may occur in the stochastic parts as explained above. Besides that, the dynamic specifications of the regression model may occur in the deterministic part, named the distributed lag model. So, general linear regression equation can be written as:

$$y_t = \sum_{i=1}^{k-2} b_i x_{ti} + b_{k-1} \sum_{j=0}^{\infty} \lambda_1^j \, x_{t-j,k-1} + b_k \sum_{j=0}^{\infty} \lambda_2^j \, x_{t-j,k} + u_t$$

where u_t is assumed to be specified by the process

$$u_t - \rho_1 u_{t-1} - \rho_2 u_{t-2} = \gamma v_{t-1} + v_t ,$$

and v_t is a pure random process. The notation ρ_1 is used for autoregressive parameters and γ is used for moving-average process parameter. λ is the parameter of the lag distribution and is assumed to be in the range $0 \leq \lambda < 1$.

The general model can be reduced to specific dynamic models under some assumptions on the parameters of λ, ρ, γ such as:

1) $\lambda_1 = \lambda_2 = \rho_1 = \rho_2 = \gamma = 0$, ordinary least squares, so that $u_t = v_t$.

2) $\lambda_1 = \lambda_2 = \rho_2 = \gamma = 0$, first order autoregressive error specification with fixed initial value. AR(1), so that, $u_t - \rho_1 u_{t-1} = v_t$,

3) $\lambda_1 = \lambda_2 = \rho_2 = \gamma = 0$, first order autoregressive error specification with stochastic initial values, $AR^*(1)$, so that, $u_t - \rho_1 u_{t-1} = v_t$,

4) $\lambda_1 = \lambda_2 = \gamma = 0$, second order autoregressive error specification with stochastic initial value, AR(2), so that $u_t - \rho_1 u_{t-1} - \rho_2 u_{t-2} = v_t$,

5) $\lambda_1 = \lambda_2 = \rho_1 = \rho_2 = 0$, first order moving average error specification, MA(1), so that $u_t = v_t + \gamma v_{t-1}$

6) $\lambda_1 = \lambda_2 = \rho_2 = 0$, first order mixed autoregressive-moving average error specification, ARMA(1,1), so that $u_t - \rho_1 u_{t-1} = \gamma v_{t-1} + v_t$,

more dynamic models depending on these parameters can be written.

7) $\rho_1 = \rho_2 = \gamma = 0$, distributed lag model with non-autorecorrelated disturbances, so that

$\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ and $u_t = v_t$,

8) $\rho_2 = \gamma = 0$, distributed lag model with first order autoregressive error specification, so that

$\lambda_1 \neq 0$, $\lambda_2 \neq 0$ and $u_t - \rho_1 u_{t-1} = v_t$,

Paseran and Slater [7].

The estimation method is based on maximum likelihood (ML) criterion as well as most of the estimation procedures currently used for all dynamic models. The ML estimators of the models with autocorrelated errors are computed using some iterative techniques.

These techniques carry out the estimation of dynamic regression models with autocorrelated errors under three different error specifications: (1) The Cochrane-Orcutt iteration procedure to compute ρ_1 for AR(1) error specification with fixed initial values, (2) the inverse interpolation to calculate the AR(1) with stochastic initial values, (2) the modified Newton-Raphson iterative technique to compute ρ_1 and ρ_2 for the AR(2) error specification with stochastic initial values [3, 4, 1].

Box and Jenkins [1], gives the ML estimators of the parameters of the regression model with first order moving-average disturbances. The derivation of the likelihood function of the ARMA process has been pursued recently by Newbold [6].

AN APPLICATION

In this chapter, a dynamic regression model of the consumption of energy was built based on per capita gross national product in 1968 retail prices in Turkey. The annual observations for the variables are shown in Table 1. Some applications on the same data was done by Koçberber [5]. Although use has tried to be made of first and second differences due to the autocorrelation in the data, nonacceptable results were obtained. In this study some dynamic regression models are tried and ML parameter estimates for different error specifications are given. The empirical work has been performed with computer programs developed by Pesaran and Slater [7]. Programs have been modified and applied to the B6800 system.

ML estimates of the parameters for dynamic models with different error specifications are given in Table 2.

Durbin-Watson test statistics, log-likelihood criterion, determination constant and error variance are shown in Table 3.

TABLE 1. Annual Data of the Variables (1951-1979)

y: Energy Consumption (Thousand tons coal equivalent)

10281	10693	11822	12251	12668	13488	14523	15172
15394	16356	16472	17790	18832	20454	21142	22907
23863	25541	27447	27910	29764	33326	36390	37703
40778	45084	49286	49767	47778			

x: Per Capita Gross National Product in 1968 Retail Prices (1000 TL)

2034.0	2218.8	2396.5	2261.5	2374.0	2390.5	2494.0
2534.1	2563.0	2576.0	2559.9	2652.8	2838.9	2882.8
2900.9	3169.2	3220.2	3349.5	3443.3	3445.8	3826.6
4015.9	4109.5	4304.1	4525.8	4784.3	4868.8	4905.8
4768.2						

Source of data: 1985 Statistical Yearbook of Turkey

TABLE 2. Error Specifications and Related Models

Model Number	Error Specification	Model
1	AR(1) Fixed Initial Values	y_t = -19924.8 + 13.75 x_t u_t = 0.50 u_{t-1} x v_t,
2	AR(1)* Stochastic Initial Values	y_t = -18629.4 + 13.43 x_t u_t = 0.55 u_{t-1} x v_t
3	A(2) Stochastic Initial Values	y_t = -18609.3 x 13.40 x_t u_t = 0.63 u_{t-1}-0.17 u_{t-2}+v_t
4	MA(1)	y_t = -18958 x 13.50 x_t u_t = 0.53 x_{t-1} x v_t
5	ARMA(1,1)	y_t = 18665 x 13.42 x_t u_t - 0.20 u_{t-1}=0.41v_{t-1}+v_t
6	Stochastically Independent Error (SIE)	y_t = -19197.1 + 13.55 x_t u_t = v_t

TABLE 3. The Test Statistics of the Models

Model No.	Error Specification	Durbin-Watson	Log-Likelihood	R^2	σ^2 Residual
1	AR(1)	1.76	-195.03	0.99	1161
2	AR(1)*	1.82	-202.84	0.99	1122
3	AR(2)	1.93	-202.49	0.99	1109
4	MA(1)	1.70	-202.84	0.99	1135
5	ARMA(1.1)	1.86	-202.67	0.99	1124
6	SIE	0.90	-207.06	0.90	1307

The determination constants, R^2, obtained for each model are very high. There is no statistically significant difference between the constants. On the other hand, it is shown that the error variances are not different from each other except in the sixth model. Hence, the log-likelihood function value is an acceptable criterion for selecting the best model. Finally, the first model was accepted as a useable model which has a maximum log-likelihood value.

CONCLUSION

As a result, a dynamic model which can be used for prediction and description is proposed at the end of the modelling process.

It has been shown that dynamic regression models must be prefered when data arises as a time series and autocorrelations cannot be removed from residuals.

ÖZET

Dışsal değişken Y_t=yıllık enerji tüketimini içsel değişken x_t=kişi başına ulusal gelir olan, yıllık veriye dayalı bir dinamik regresyon modeli kurulmuştur. Hata terimini AR(1), AR(2), MA(1), ARMA(1,1) durağan stokastik süreçlere uyduğu dinamik regresyon modelleri incelenmiş ve bunların en çok olabilirlik tahmin edicileri tanıtılmıştır. Sonuçta, dinamik regresyon modellerinin klasik regresyon modellerine göre daha etkin olduğu görülmüştür.

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APPENDIX

Hacettepe Bulletin of Natural Sciences and Engineering
1985-1986/ Volume 14-15/pp. 109-112

PREPERATION OF FINAL TYPESCRIPT*
- APPLIED AND EXPERIMENTAL SCIENCES -

E. Board[1]

This article gives instructions for preparing the final typescript of your paper. It is laid out according to these rules, and may be used as a guide. Please indent the abstract three spaces from the left and right margins, as shown here.

Key words: Type styles, Layout, Spacing

INTRODUCTION

The rules below amplify the information given in [1], and aim at giving the **bulletin** a uniform and pleasing appearance. Please follow them carefully. The Editorial Board are under no obligation to publish typescripts not conforming to these rules. Please note that the page numbers and publication identifier (top of page one) will be added during publication, but you should number the pages **well outside the printing area**, so they may be kept in order.

MATERIALS AND METHODS

The final typescript should be prepared on an electric typewriter employing a black plastic ribbon and one of the following typefaces

(a) For the main text:

IBM letter Goth 96 or OLYMPIA 808

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(1) Hacettepe University, Faculty of Science, Ankara, TURKEY.

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(b) For the abstract, references, footnotes, subscripts, and superscripts:

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12 Pitch (12 characters per inch) **should be used throughout.** The main title and subheadings should be in upper case. Bold face type may be used to highlight newly defined terms, important phrases, etc. **Do not use underlining.** Special symbols, foreign letters, etc., should be typed whenever possible.

The typing area is 14cmx22cm, giving 52 lines and 66 characters per line at 12 pitch. **Only material within this area will appear in print.**

The main title and subheadings should be centred. Displayed formulae, etc., may also be centred if desired.

There should be an indentation of three spaces at the beginning of each new paragraph. Part identifiers, such as (a), (b), etc., should also be indented three spaces from the left hand margin.

The (first line of the) main title should be typed on line 8, the authors name(s) on line 14 and the first line of the abstract on line 17. Where there is more than one author the format for the names is A. Abel$^{(1)}$, B. Cox$^{(\cdot)}$, ... After the first page the text should begin on line 3. The following table gives the rules for line spacing.

TABLE 1. Line Spacing.

Spacing	Application
1/2	Subscripts, superscripts.
1	Abstract. Two lines of the same reference or footnote
1 1/2	Normal text. Between two references or footnotes. Double-lined title or heading.
2	Between text and heading to a table or figure. To emphasize a formula or block of text.
2 1/2	Between a subheading and following text or item.
3 1/2	Between text and following subheading.

SUBSECTIONS

The manuscript should begin with an abstract and introduction, and end with a Turkish summary (özet) and references. The main body of the text should normally be collected into unnumbered sections headed materials and methods, results, discussion. However other headings may be used if the above are inappropriate.

TABLES AND FIGURES

Number tables and figures independently and consecutively throughout the paper. Table headings should be placed above the table and consist of the word "table" in upper case, the table number and a short caption. The caption should be in lower case with the first letter of each noun in upper case. Figures should be dealt with in a similar way, but with the heading and caption below. See Table 1 and Figure 2 below.

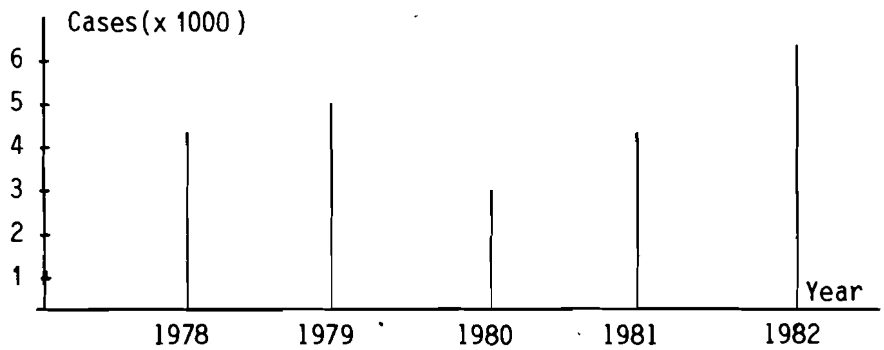

FIGURE 1. History of Virus "A" Infection.

Any hand work should be drawn carefully using indian ink. Where figures are prepared on seperate sheets they should be fixed carefully to the typescript in the correct place. Any lettering should use the same typeface as the text. Use only horizontal dividing lines in tables. Internal spacing of tables and figures is left to the discretion of the author. Wherever possible tables and figures should be designed to fit neatly between the side margins. However very long tables or figures may, together with their headings, be set lengthwise along the page. A page may contain more than one figure or table set lengthwise, **but on no account should it contain any text.**

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(b) References. The punctuation of references is given inside the back cover of the **bulletin**, to which reference should be made.

(c) Acknowledgments. Personal acknowledgments may be placed just before the Turkish Summary (Özet), as shown below.

Acknowledgment. The author would like to thank ...

ÖZET

Bu makale, yazınızın son şeklinin hazırlanışı ile ilgili kuralları içermektedir. Aynı zamanda kendisi bu kurallara göre hazırlandığı için bir örnek teşkil etmektedir.

REFERENCES

1. Board, E. Submission of manuscripts. Hacettepe Bulletin of Natural Sciences and Engineering, 14, Inside back cover, 1985.

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1980 Subject Classification: 00A20

1. INTRODUCTION

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12 Pitch (12 characters per inch) **should be used throughout.** The main title, subheadings and numbered section headings should be in upper case. Bold face type may be used to highlight newly defined terms, important phrases, etc. **Do not use underlining.** Mathematical symbols, foreign letters, etc., should be typed whenever possible. Where script letters are called for, but not available, please use the nearest typeable alternative (eg. upper case italic or boldface).

3. LAYOUT AND SPACING

The typing area is 14cmx22cm, giving 52 lines and 66 characters per line at 12 pitch. **Only material within this area will appear in print.**

The main title and subheadings should be centred. Displayed formulae, etc., may also be centred if desired.

3.1. DEFINITION. Text which forms part of a numbered item (remark, definition, statement or proof of a theorem, etc.) will be called **bound**, all other text will be called **free.**

There should be an indentation of three spaces at the beginning of each new paragraph. Each new block of free text should begin with a paragraph indentation, but bound text should not contain paragraph indentations. However the word "proof", and part identifiers such as (a), (b), etc., should be indented three spaces. An example is seen in the layout of Theorem 3.2 below .

3.2. THEOREM. let (x_n) be a convergent sequence in the Hausdorff space X. Then:

(a) Every subsequence of (x_n) is convergent.

(b) The limit of (x_n) is unique.

Proof. (a) Immediate from the definition.

(b) Suppose $x \neq y$ are limits of (x_n). Then if M and N are arbitrary neighbourhoods of x, y respectively .....

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3.3. TABLE. Line Spacing.

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2	Between statement and proof of a theorem, etc. Between bound and free or bound and bound text.
2 1/2	Between a subheading and following text or item.
3 1/2	Between text and following subheading.

Where subscripts, superscripts, etc., are involved the above spacing may be increased the minimum necessary to maintain clarity.

3.4. EXAMPLE. Increased Spacing.

(a) $\ldots x_a \ldots\ldots$
$\ldots\ldots 2^n \ldots$

(b) $\ldots\ldots x_a \ldots$
$\ldots 2^n \ldots$

Try to avoid case (b) by rearranging the text. Also avoid the use of subscripted subscripts, etc.

4. MISCELLANEOUS NOTES

(a) Tables, Diagrams, Graphs, etc. These should be treated as numbered items. If prepared on seperate sheets they should be fixed to the typescript in the correct place. Any lettering should use the same typeface as the text.

(b) Displayed Formulae. These may be given a number on the right. Give a two line spacing above and below. See the example below.

4.1. EXAMPLE. Displayed Formula.
If A is a real square matrix then the inverse A^{-1} is given by

$$A^{-1} = \frac{1}{|A|} \, \mathrm{Adj}(A) \qquad \ldots(4.1)$$

provided $|A| \neq 0$.

Display the elements of a matrix between square brackets. Very large matrices may be typed with a finer pitch and the smaller of the two typefaces, if necessary.

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